Stockton University
Mathematical Mayhem 2019
Group Round - Solutions

March 30, 2019

Name: ____

Name: ____

Name: ____

High School:____

Instructions:

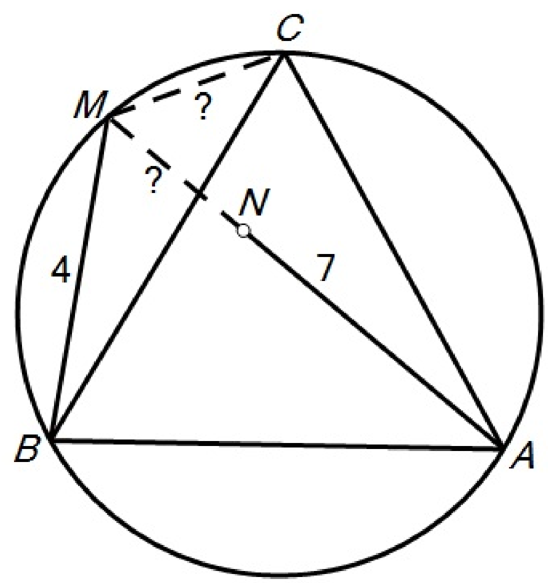
- This round consists of **5** problems worth **16** points each for a total of **80** points.
- Each of the 5 problems is free response.
- Write your complete solution in the space provided including all supporting work.
- No calculators are permitted.
- This round is **75 minutes** long. **Good Luck!**

OFFICIAL USE ONLY:

Problem #	1	2	3	4	5	Total
Points Earned						

♠ Group Round ♠

Problem 1. Consider the diagram below in which equilateral triangle $\triangle ABC$ is inscribed in a circle. The point M is on the arc of the circle somewhere between points B and C so that the distance between A and M is 7 inches and the distance between B and M is 4 inches. What is the distance between C and M? **The diagram below is not drawn to scale.**



Solution to Question 1.

On the line AM, pick the point N such that $MC = MN$. Observe that $\angle CMA$ and $\angle CBA$ intercept the same arc, so $\angle CMA = \angle CBA = 60°$. Therefore, $\triangle CMN$ is an equilateral triangle. Next, $\triangle ACN$ is congruent to $\triangle MCB$, since $AC = BC$, $CN = CM$, and $\angle ACN = \angle ACM - 60° = \angle MCB$. Therefore $BM = AN$, and further $AM = AN + MN = BM + MC$. This gives $CM = AM - BM = 7\,\text{in} - 4\,\text{in} = \boxed{3\,\text{in}}$.

Problem 2. Assume that you have two identical cups. The first is full of water, while the second is empty. You pour half of the water from the first cup into the second. Then on the second transfer, you pour one third of the water in the second cup back into the first. On the third transfer, you pour one fourth of the water in the first cup back into the second cup. You repeat this pattern, alternating cups; on the nth transfer you pour $1/(n+1)$ of the water that is in one cup back into the other cup. What fraction of the water is in the first cup just **before** the 19th transfer?

Solution to Question 2.

Note that before the 3rd transfer, the fraction of the water in the first cup is:

$$\frac{1}{1} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{3} = \frac{2}{3}.$$

Before the 5th transfer, the fraction of the water in the first cup is:

$$\frac{2}{3} \times \frac{3}{4} + \frac{1}{5}\left[\left(\frac{1}{1} - \frac{2}{3}\right) + \left(\frac{2}{3} \times \frac{1}{4}\right)\right] = \frac{1}{2} + \frac{1}{5}\left[\frac{1}{3} + \frac{1}{6}\right] = \frac{3}{5}.$$

Before the 7th transfer, the fraction of the water in the first cup is:

$$\frac{3}{5} \times \frac{5}{6} + \frac{1}{7}\left[\left(\frac{1}{1} - \frac{3}{5}\right) + \left(\frac{3}{5} \times \frac{1}{6}\right)\right] = \frac{1}{2} + \frac{1}{7}\left[\frac{2}{5} + \frac{1}{10}\right] = \frac{4}{7}.$$

This suggests a pattern, that after the nth transfer, the fraction of the water in the first cup is $\dfrac{(n+1)/2}{n}$. To show this, we use induction, taking any of the above examples as our base case. Let A_n be the fraction of the water in the first cup before the n-th transfer and assume $A_{n-2} = \frac{(n-2)+1}{2(n-2)}$. Then

$$\begin{aligned}
A_n &= \frac{(n-2)+1}{2(n-2)} \times \frac{n-2}{n-1} + \frac{1}{n}\left[\left(\frac{1}{1} - \frac{(n-2)+1}{2(n-2)}\right) + \left(\frac{(n-2)+1}{2(n-2)} \times \frac{1}{n-1}\right)\right] \\
&= \frac{1}{2} + \frac{1}{n}\left[\frac{2(n-2)-n+1}{2(n-2)} + \frac{1}{2(n-2)}\right] \\
&= \frac{1}{2} + \frac{1}{n}\left[\frac{2n-4-n+2}{2(n-2)}\right] \\
&= \frac{1}{2} + \frac{1}{n}\left[\frac{n-2}{2(n-2)}\right] \\
&= \frac{1}{2} + \frac{1}{2n} = \frac{n+1}{2n}
\end{aligned}$$

So, we seek A_{19} which equals $\dfrac{19+1}{2(19)} = \dfrac{20}{38} = \boxed{\dfrac{10}{19}}$.

Problem 3. Five consecutive positive integers have the property that the sum of the second, third, and fourth is a perfect square, while the sum of all five is a perfect cube. If m is the third number in the list, what is the smallest possible value of m?

Solution to Question 3.

Since m is the third of the five integers, then the five integers are $m - 2$, $m - 1$, m, $m + 1$, and $m + 2$. The sum of all five is thus $(m - 2) + (m - 1) + m + (m + 1) + (m + 2) = 5m$ and the sum of the middle three is $(m - 1) + m + (m + 1) = 3m$. Therefore, we want to find the smallest integer m for which $3m$ is a perfect square and $5m$ is a perfect cube. Consider writing m, $3m$, and $5m$ each as a product of prime numbers. For $3m$ to be a perfect square, each prime must occur an even number of times in the product. Thus, the prime 3 must occur an odd number of times in the product that represents m. For $5m$ to be a perfect cube, the number of times that each prime occurs in the product must be a multiple of 3. Thus, the prime 5 must occur a number of times which is one less than a multiple of 3 in the product that represents m. Since both of the primes 3 and 5 are factors of m, then to minimize m no other prime should occur. For $3m$ to be a perfect square, 5 must occur an even number of times in $3m$. For $5m$ to be a perfect cube, the number of times that 3 occurs in $5m$ is a multiple of 3. Therefore, m is a number which contains an odd number of 3s and an even number of 5s (since $3m$ does), and at the same time contains 3 a number of times which is a multiple of 3 (since $5m$ does) and 5 a number of times that is 1 less than a multiple of 3. To minimize m, m should contain as few 3s and 5s as possible, so should contain three 3s and two 5s, so $m = 3^3 5^2 = \boxed{675}$.

Problem 4. Given a collection of numbers *a*, *b*, *c*, and *d*, an *algebraic expression* made from the numbers is any expression

- that uses only the numbers *a*, *b*, *c*, and *d* and uses each of these numbers exactly once;
- that can use the numbers *a*, *b*, *c*, and *d* as any digit within a base 10 number;
- that only uses the following operations: + (addition), $-$ (subtraction), $\times$ (multiplication), $\div$ (division), and $\wedge$ (exponentiation); and
- that can use parentheses.

For example, here are some algebraic expressions made from the numbers 1, 9, 7, and 3:

$$97 \times (3-1) \qquad 7^{19+3} \qquad (93+7) \div 1$$
$$9731 \qquad 17^{93} \qquad (1+9+7) \times 3$$

What is the largest number that can represented by an algebraic expression using the numbers

$$2, 0, 1, \text{ and } 9?$$

Solution to Question 4.

Intuitive Version, for students: Both division and subtraction can be disregarded, as there are always ways to makes expressions that are equal or greater than those made by using division and subtraction by replacing those operations with multiplication and addition respectively. So we focus on addition, multiplication, and exponentiation. Of those, a greater result can generally be achieved using exponentiation (unless 1 is involved, then adding one is the best we can do), so we will only look at expressions that contain only exponentiation. We'll sort our possible answers by the number of digits that are in the numbers that they use. Specifically,

- *One four digit numbers*: 9210 is the largest possible four digit number.
- *Four one digit numbers*: There is no effective manner to use 0 to increase our value. So, we will add 1. By the discussion above, the four possible considerations are 2^{9+1}, $(2+1)^9$, 9^{2+1} and $(9+1)^2$. The largest is $(2+1)^9 = 19683$.
- *One three digit number and one one digit number*: We sort these possibilities by the one digit number. Leaving 0 by itself gives 921 as the largest possible expression. Leaving 1 by itself gives $1 + 920 = 921$. Leaving 2 by itself yields 2^{910} and 910^2. The first of these is much larger since $910 < 1024 = 2^{10}$ and so $910^2 < 2^{20} < 2^{910}$. Leaving the 9 by itself gives 9^{210} and 210^9. Since $210 < 729 = 9^3$ we have $210^9 < (9^3)^9 = 9^{27} < 9^{210}$. Hence 9^{210} and 2^{910} are the largest possibilities, and since $9 < 16 = 2^4$, we see that $9^{210} < (2^4)^{210} < 2^{840} < 2^{910}$. So, 2^{910} is the largest possible algebraic expression with one three digit number and one one digit number.
- *Two two digit numbers*: There are 3 possible pairs of two digit numbers. So, there are 6 values to consider:

  $$10^{92}, 92^{10}, 20^{91}, 91^{20}, 21^{90}, 90^{21}.$$

  Each of these is less than 2^{910}, since the largest base is less than $128 = 2^7$ and the largest exponent is 92. Each of these is less than $(2^7)^{92} = 2^{644}$.

- *One two digit number and three one digit numbers*: Note that if 0 is one of the one digit numbers, it cannot be used to increase a value. So every algebraic expression that can be made from one two digit number and one one digit number from 9, 2, and 1 is less than or equal to an algebraic expression that can be made by two two digit numbers. The only possible trios of numbers to consider are: $2, 9, 10; 1, 9, 20;$ and $1, 2, 90$. Using $1, 2$, and 90, the largest possible values are $(1+2)^{90}, 2^{90+1}, (90+1)^2$, and 90^{2+1}. Using $1, 9$, and 20, the largest possible values are $(1+9)^{20}, 9^{20+1}, (20+1)^9$, and 20^{9+1}. Each of these is less than 10^{92} or 92^{10}, both of which are less than 2^{910}.

  The last possibilities are the most complicated. Consider $2, 9$, and 10. The largest values can be made by making an exponent tower using these numbers. There are 6 possible orders for these three numbers, and two ways to put parentheses into these expressions. For example, compare $9^{2^{10}}$ to $(9^2)^{10}$. Note that $(9^2)^{10} = 9^{20} < 9^{2^{10}}$. Performing this comparison to the 6 possible orders, we have 6 candidates for the largest algebraic expression using 2, 9, and 10. These are

  $$2^{10^9}, 2^{9^{10}}, 9^{2^{10}}, 9^{10^2}, 10^{9^2}, \text{ and } 10^{2^9}.$$

  Note that $100 = 10^2 < 2^{10} = 1024$, $81 = 9^2 < 2^9 = 512$ and $9^{10} > 10^9$. To see this last one, observe that $9^2 = 81 > 80 = 2^4(5)$, and

  $$9^{10} = (9^2)^5 > (2^4(5))^5 = 2^{20}5^5 = \frac{2^{11}}{5^4}(2^9 5^9) = \frac{2048}{625}10^9.$$

  Further, $9^{10} > 910$ and so there are three candidate values remaining: $2^{9^{10}}, 9^{1024}$, and 10^{512}. Notice that both 9 and 10 are less than 2^4. So, $9^{1024} < (2^4)^{1024} = 2^{4096}$ and $10^{512} < (2^4)^{512} < 2^{4096}$. But $9^4 = 6561 > 4096$ and so $2^{9^{10}} > 2^{9^4} > 2^{4096}$.

Hence, $\boxed{2^{9^{10}}}$ is the largest algebraic expression that can be made from the digits $2, 0, 1$, and 9.

Details, for Teachers: We make a few assumptions in the intuitive solution for students that require justification.

Both division and subtraction can be disregarded: There are always ways to makes expressions that are equal or greater than those made by using division and subtraction by replacing those operations with multiplication and addition respectively. Here are the details of the argument:

- Subtraction is used and division is not: in this case, replacing each instance of subtraction with addition will produce a larger value. This claim can be verified by testing all of the possible numbers of times that subtraction could be performed (1, 2, or 3), and seeing that addition is an improvement in each case.

- Division is used and subtraction is not: Since subtraction is not used, no non-negative values can be made. Note, if $a \div b > a \times b$ and $a, b > 0$, then $b < 1$; so if division is used only once it can be replaced by multiplication to create a larger expression. Any attempts to perform division consecutively without an operation in the middle could handled by multiplying once and dividing once since $a \div (b \div c) = (a \times c) \div b$ and $(a \div b) \div c = a \div (b \times c)$. Note that $(a \div b) \times (c \div d) = (a \times d) \div (b \times c)$, $(a \div b) + (c \div d) \leq (a \times b) + (c \times d)$ and $(a \div b)^{c \div d} \leq (a \times b)^{c \times d}$. So, the only case left in which division can occur absent of subtraction is as the first and last of three operations performed. A careful consideration of these cases shows that either an instance of division can be for multiplication or using multiplication instead of division increases the final value of the algebraic expression.

- Division and Subtraction are both used: There are many different possible combinations of operations and parentheses that fall under this category. When working with 4 values and 3 operations, there are 72 combinations of operations and parentheses that include one division and one subtraction. In each case, larger values can be obtained through addition and multiplication.

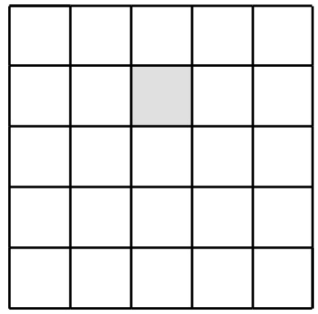
Ignoring subtraction and division, we consider only addition, multiplication, and exponentiation. This means we are working only with non-negative integers.

There is no advantage in making smaller values: These algebraic expressions contain 0, 1, 2, or 3 operations. Note that the 3 operations $A(x, y) = x + y$, $E(x, y) = x^y$ and $M(x, y) = xy$ all increase on $[1, \infty)$ for both x and y. That is, the larger of a number that we can create with each operation individually, the larger our number will be. For example, $(3+2)^{4+5} < (3 \times 2)^{4+5} < (3 \times 2)^{4\times 5}$ since $3+2 < 3 \times 2$ and $4+5 < 4 \times 5$. There is no advantage in creating a smaller number at any point in the expression.

Exponentiation gives answers that are larger than addition or multiplication: Note that if a and b are both greater than or equal to 2 and $a \geq b$, then

$$
\begin{aligned}
a^b &\geq a^2 \geq a(a) \geq ab > a/b, \\
a^b &\geq a^2 \geq a(a) \geq 2a \geq a+b > a-b.
\end{aligned}
$$

Problem 5. Consider the 5×5 grid, which is made up of 6 vertical and 6 horizontal grid lines, shown below. The square in the 2nd row and 3rd column of the grid is shaded.



Many squares can be made using these grid lines. The shaded square is contained in 16 of the squares that can be made using the grid lines.

(A) In a 20×20 grid, there are 42 grid lines. How many squares made using the grid lines contain the square that is in the 5th row and 8th column of the grid? **Simplify your answer.**

(B) In an $n \times n$ grid, there are $2(n+1)$ grid lines. How many squares made using the grid lines contain the square that is in the 5th row and 8th column of the grid? **You may assume** $n > 16$**.**

Solution to Question 5.

(A) Let A_k denote the number of $k \times k$ squares containing the square in the 5-th row and 8-th column. Then

$$A_k = \begin{cases} k^2 & \text{if } k \leq 5, \\ 5(k) & \text{if } 5 < k \leq 8, \\ 5(8) & \text{if } 8 < k \leq 12, \\ 5(21-k) & \text{if } 12 < k \leq 15, \text{ or} \\ (21-k)^2 & \text{if } 15 < k \leq 20. \end{cases}$$

This yields the following sum

$$\begin{aligned} \sum_{k=1}^{20} A_k &= \sum_{k=1}^{5} k^2 + \sum_{k=6}^{8} 5(k) + \sum_{k=9}^{12} 5(8) + \sum_{k=13}^{15} 5(21-k) + \sum_{k=16}^{20} (21-k)^2 \\ &= \frac{5(6)(11)}{6} + 5(6+7+8) + 4(5)(8) + 5(6+7+8) + \frac{5(6)(11)}{6} \\ &= 55 + 105 + 160 + 105 + 55 \\ &= \boxed{480} \end{aligned}$$

(B) Generalizing our equations from part (A), let A_k denote the number of $k \times k$ squares that containing the square in the 5-th row and 8-th column. Then:

$$A_k = \begin{cases} k^2 & \text{if } k \le 5, \\ 5k & \text{if } 5 < k \le 8, \\ 5(8) & \text{if } 8 < k \le n-8, \\ 5(n+1-k) & \text{if } n-8 < k \le n-5, \text{ or} \\ (n+1-k)^2 & \text{if } n-5 < k \le n. \end{cases}$$

This yields the following sum:

$$\begin{aligned} \sum_{k=1}^{n} A_k &= \sum_{k=1}^{5} k^2 + \sum_{k=6}^{8} 5(k) + \sum_{k=9}^{n-8} 5(8) + \sum_{k=n-7}^{n-5} 5(n-k+1) + \sum_{k=n-4}^{n} (n-k+1)^2 \\ &= \frac{5(5+1)(2(5)+1)}{3} + 10(6+7+8) + (n-16)(5)(8) \\ &= 320 + 40n - 640 \\ &= \boxed{40n - 320} \end{aligned}$$