Rubik's Cubes and Group Theory

Presented by : Sean Harrell
The Richard Stockton College of New Jersey
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Project Advisor: Dr. Bradley Forrest

How it began…

- It began as a piqued interest at the 2011 Moravian conference
- Sought answers to 2 questions:
  - Is there a mathematical way to “map” a Rubik’s Cube?
  - How do we determine Valid and Invalid Configurations on the Rubik’s Cube?

Moves

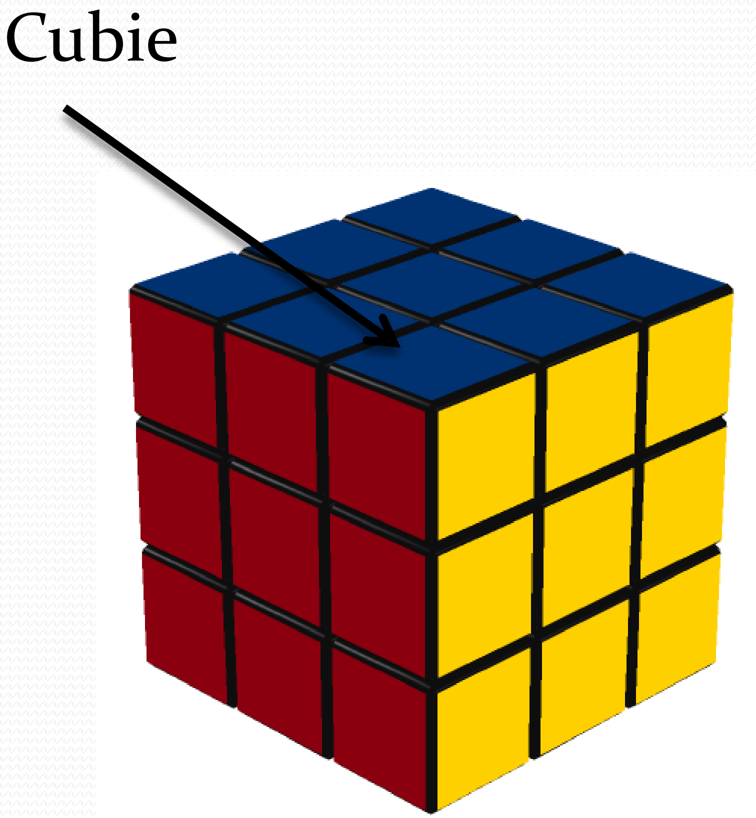
- All moves on the Rubik’s cube can be expressed in six basic moves
- A move is a single 90° clockwise rotation
  - U - Up
  - D - Down
  - F - Front
  - B - Back
  - L - Left
  - R – Right

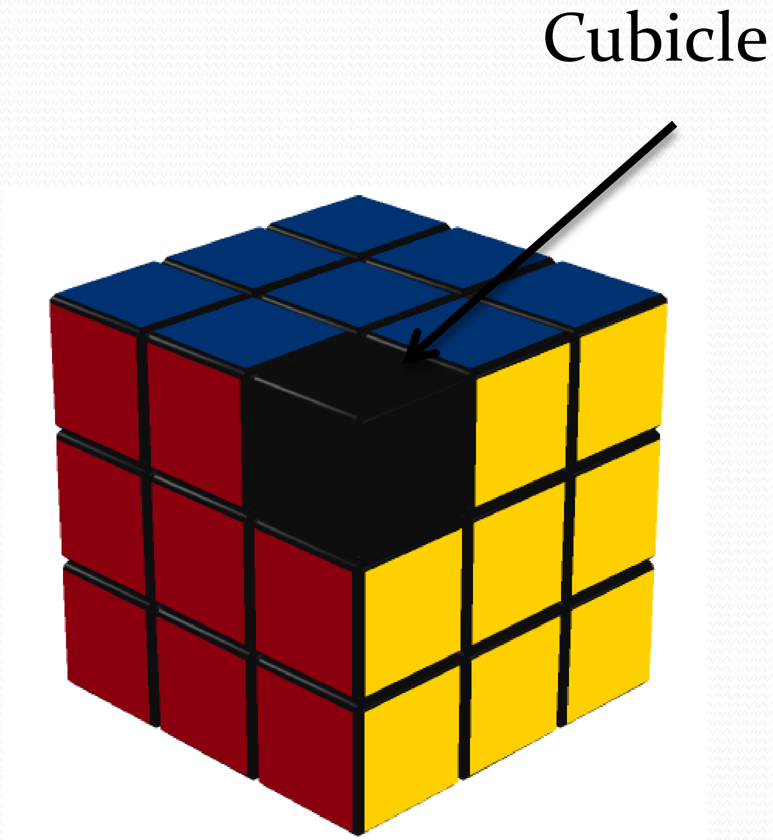
The Structure of the Cube

- To understand the structure of the cube we need to define two terms:

1. Cubie – the freely moving pieces on the Rubik's cube
2. Cubicle – the "core" non-moving positions where the cubies belong in standard or solved position

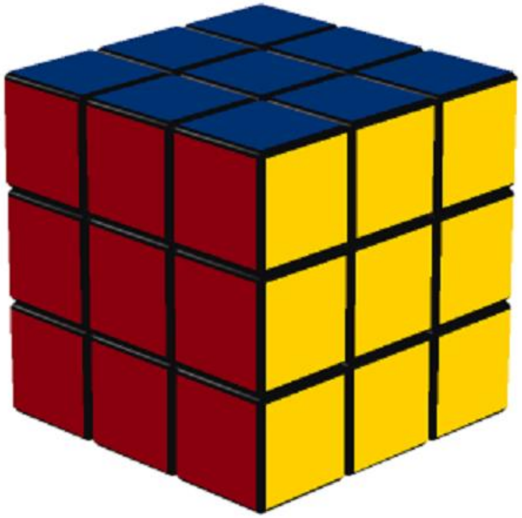
The Structure of the Cube

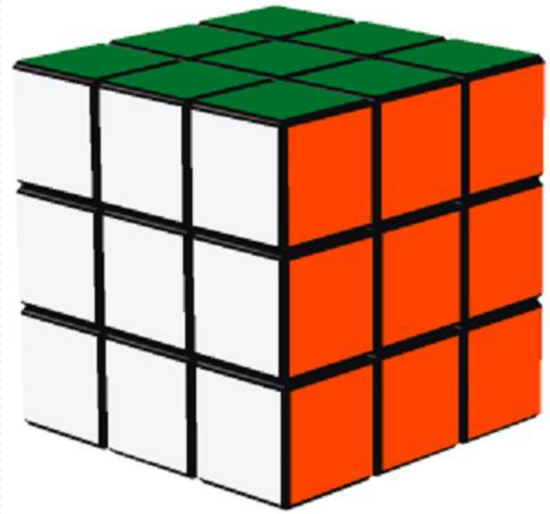




Cubies

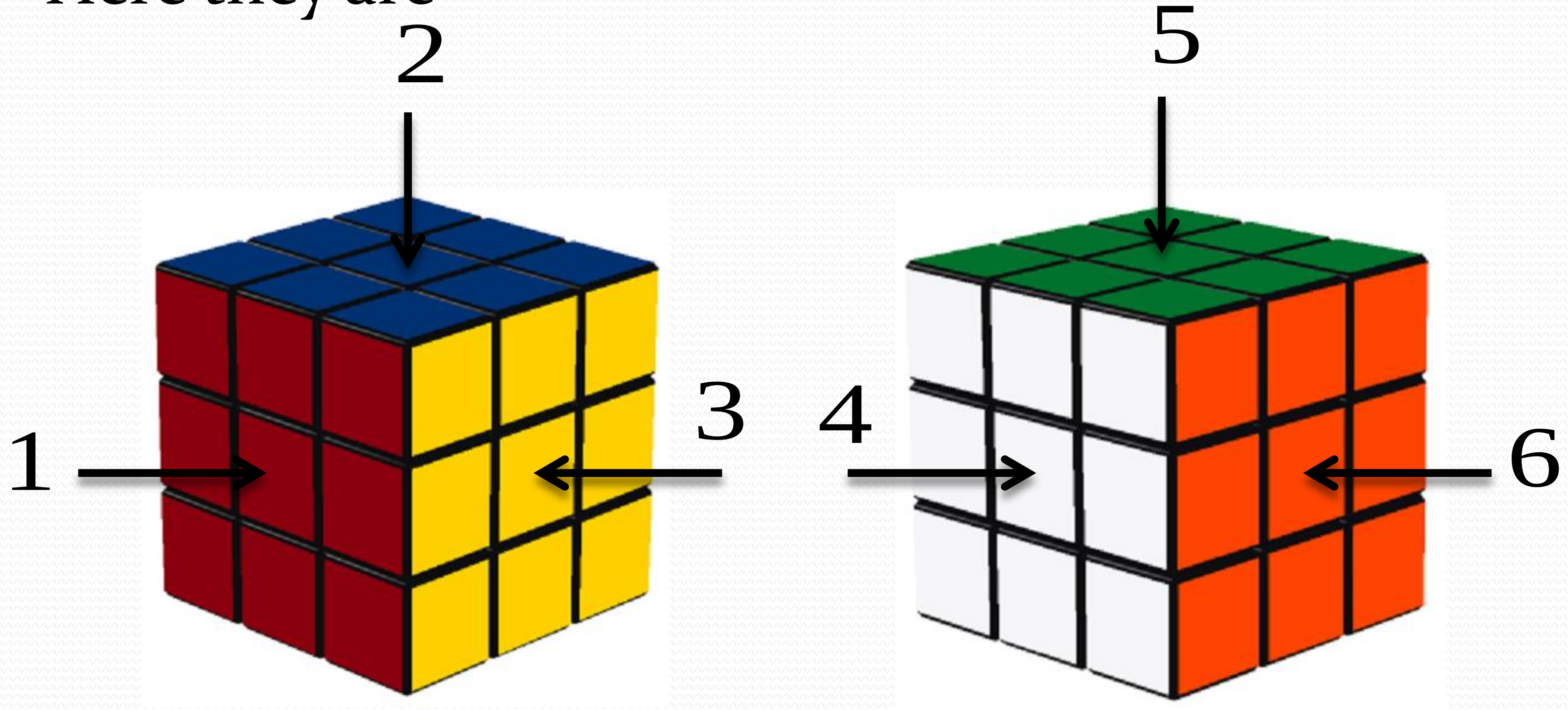
- There are six center cubies





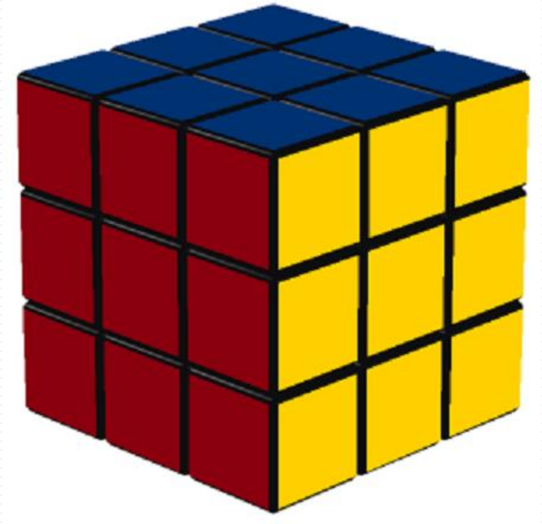
Cubies

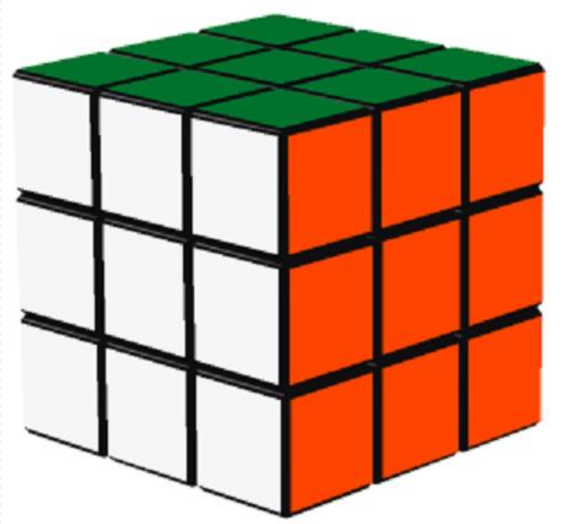
- Here they are



Cubies

- There are 8 corner cubies





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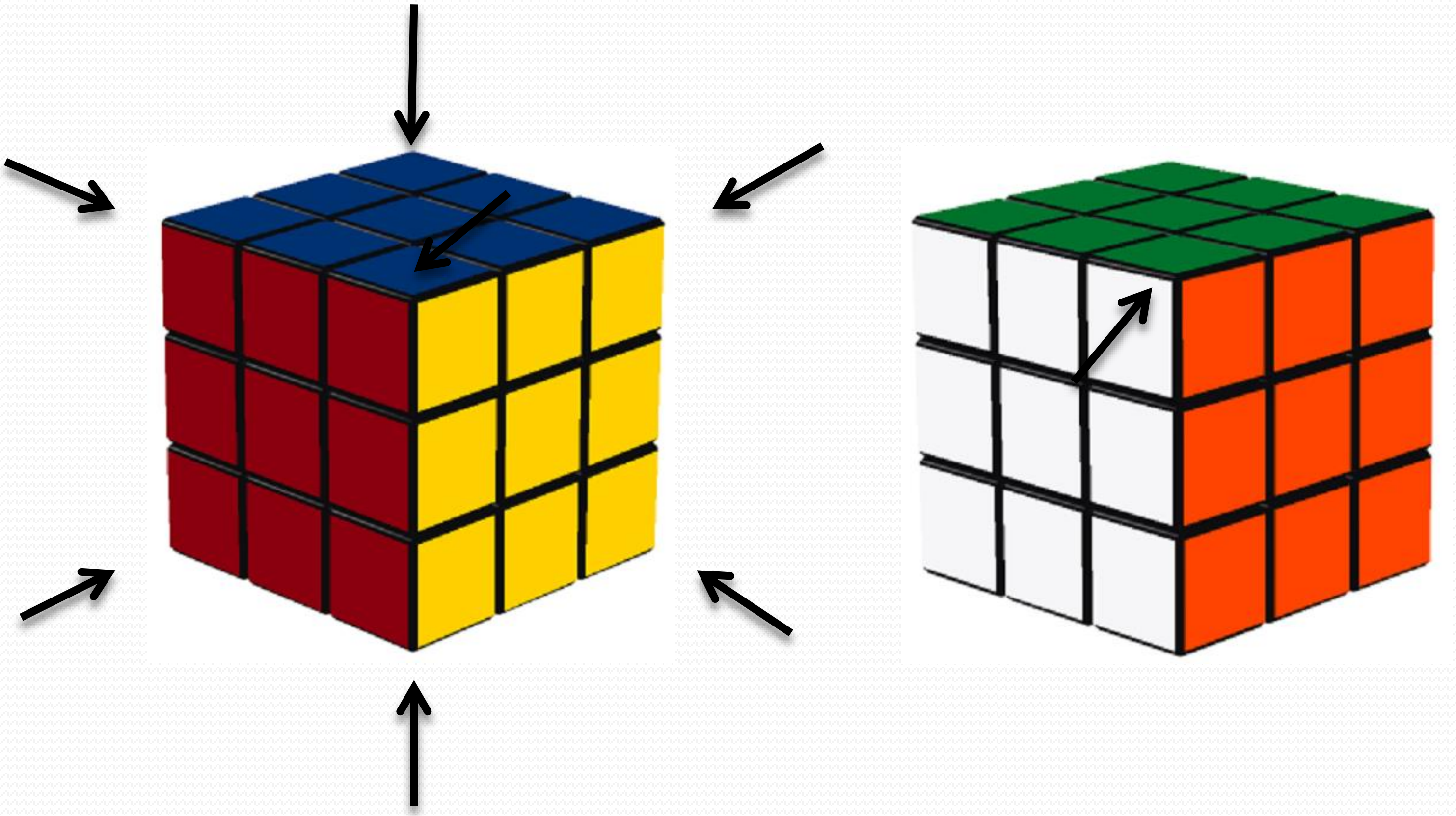
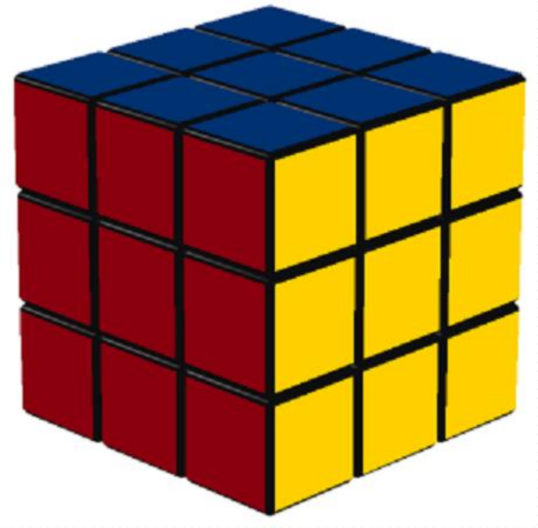
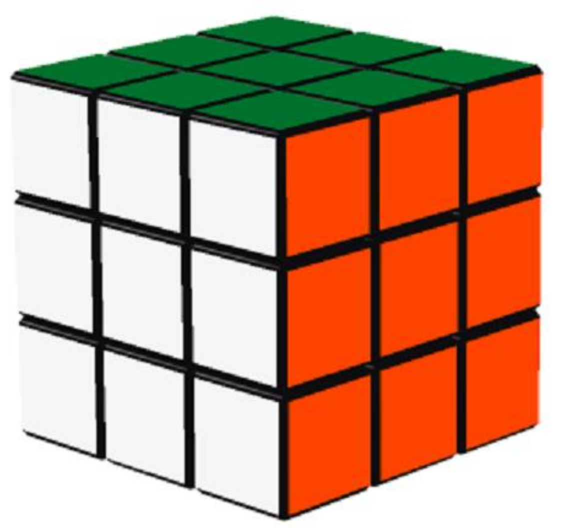


Cubies

- We number the corner cubies as follows:





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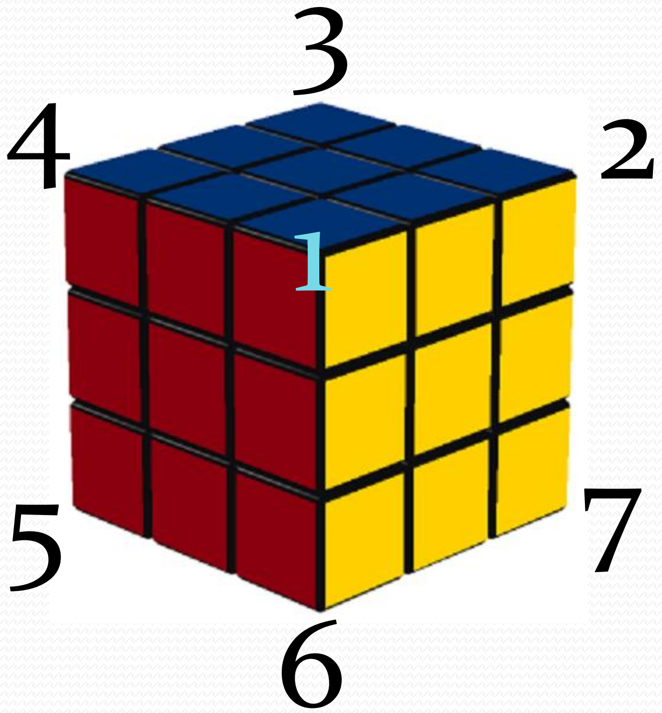
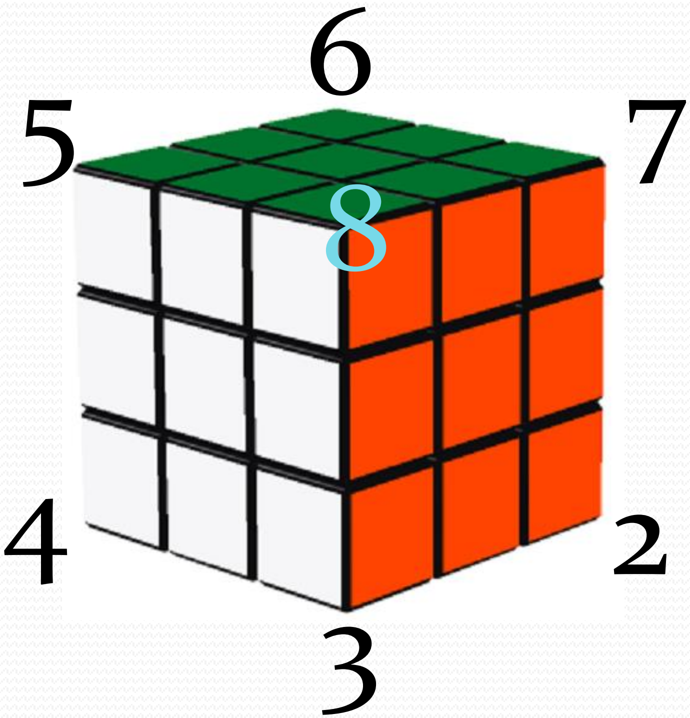




Cubies

- There are 8! ways to rearrange corner cubies.

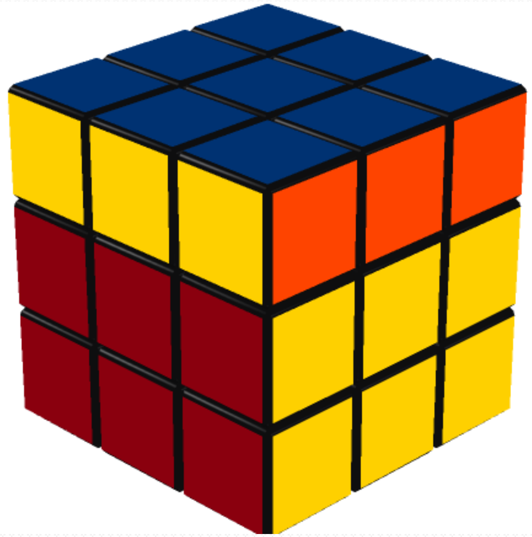
- Each corresponds to a **permutation** of the form:

  (1, 2, 3, 4, 5, 6, 7, 8)

Cubies

The images below correspond to an **upper or U twist** on the Rubik's cube
The U twist permutation has the form:

(1, 2, 3, 4, 5, 6, 7, 8)
↓
(2, 3, 4, 1, 5, 6, 7, 8)



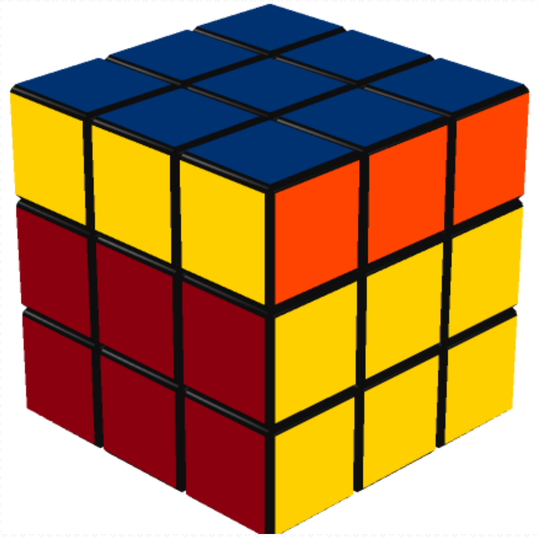
Cubies

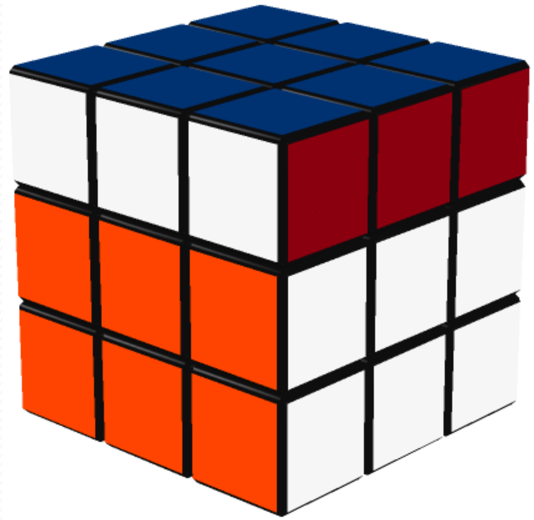
We then take the permutation

$$(2, 3, 4, 1, 5, 6, 7, 8)$$

and express it in the **cyclic form**:

$$\boldsymbol{\sigma} = (1\ 4\ 3\ 2)\ (5)\ (6)\ (7)\ (8) = \mathbf{(1\ 4\ 3\ 2)}$$





Cubies

- As a result of function composition:

$$(1\ 4\ 3\ 2)=(14)(13)(12)$$

- We will find later that for every n, its n-cycle can be written as either an even or an odd number of permutations

Cubies

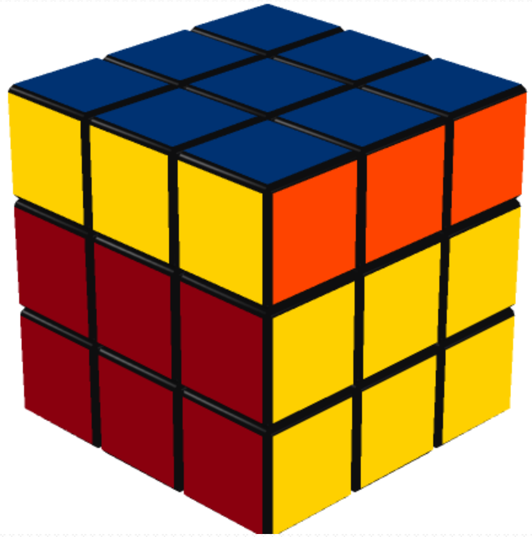
Some powers of U

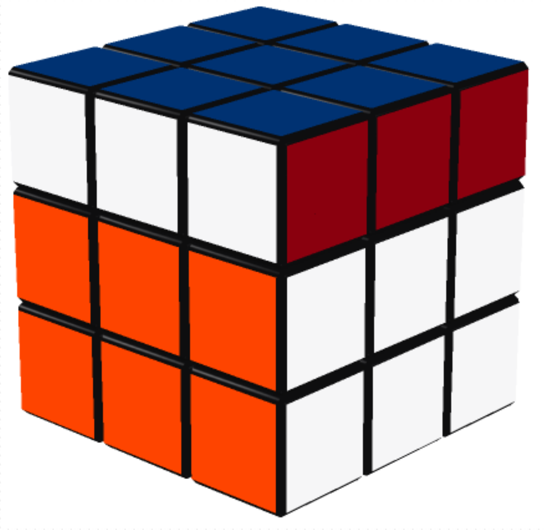
U: σ = (1 4 3 2)

U^2: σ = (13) (24)

U^3: σ = (1 2 3 4)

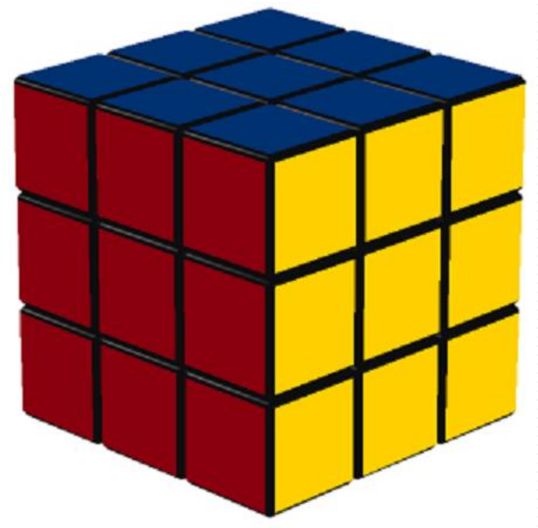
U^4: σ = (1)(2)(3)(4) = 1

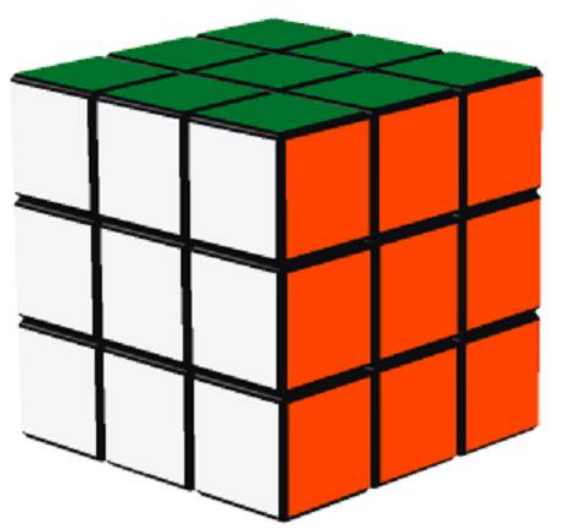




Cubies

- There are 12 edge cubies





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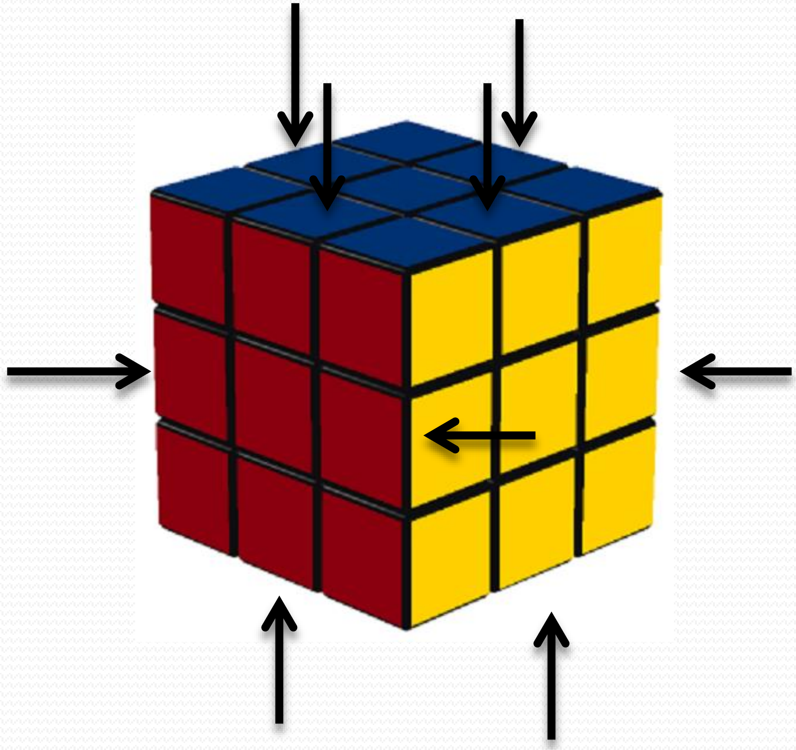

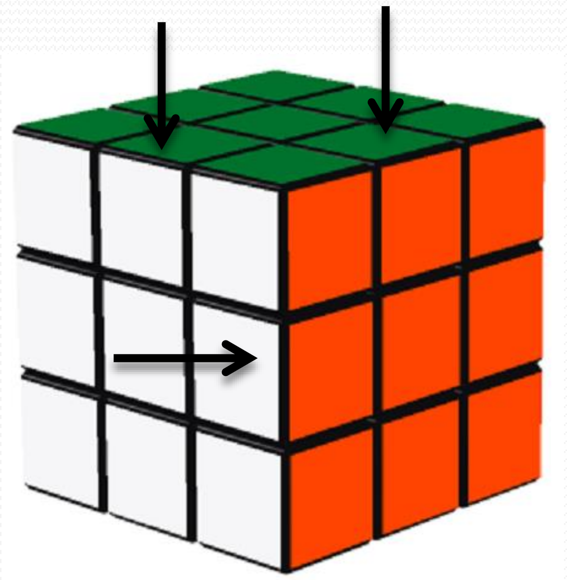

Cubies

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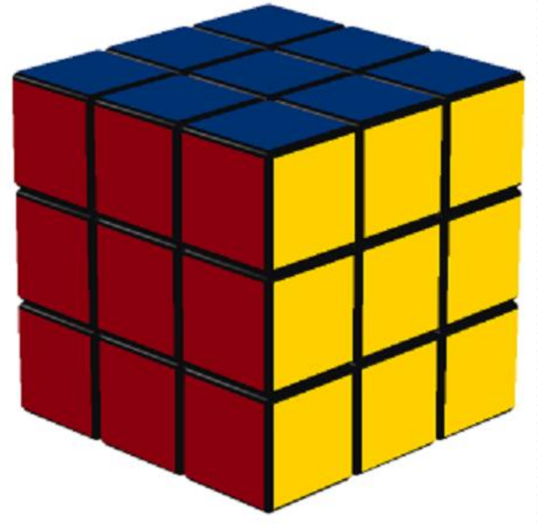

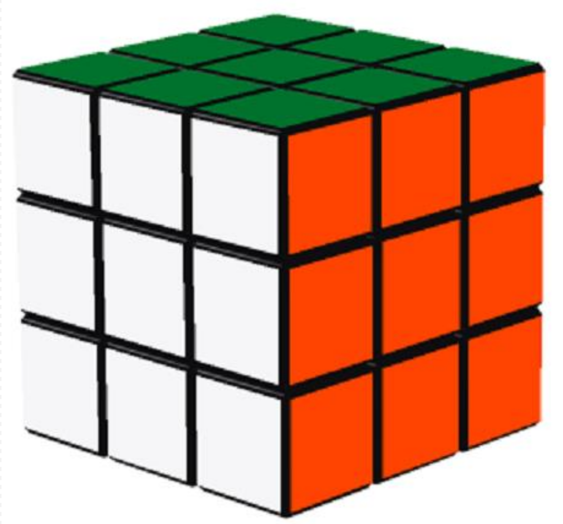

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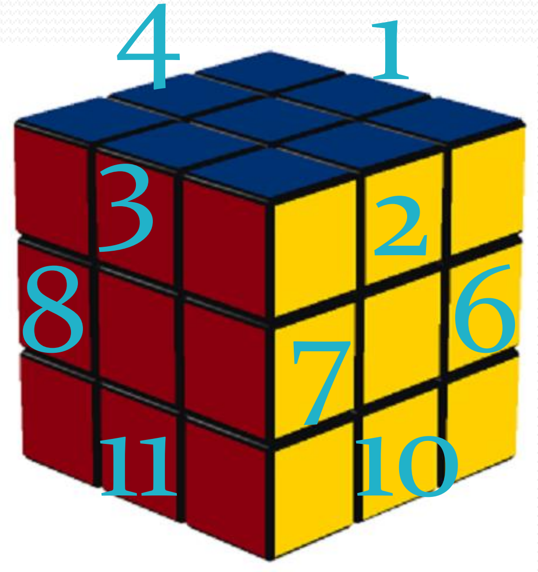
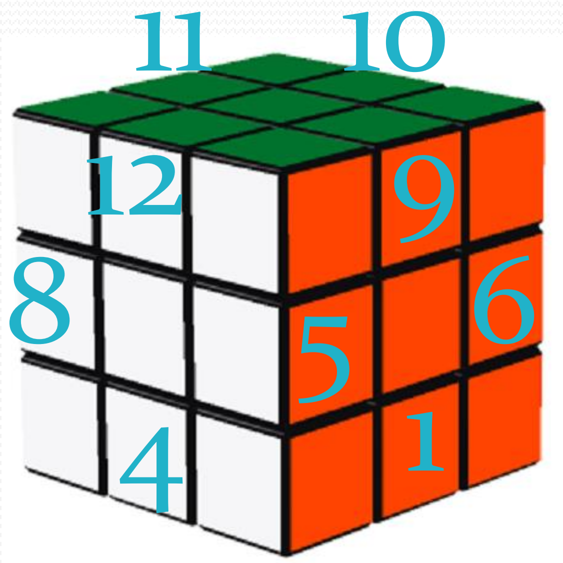




Cubies

- There are 12! ways to rearrange the edge cubies.
- Each corresponds to a **permutation** of the form:

$$(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12)$$

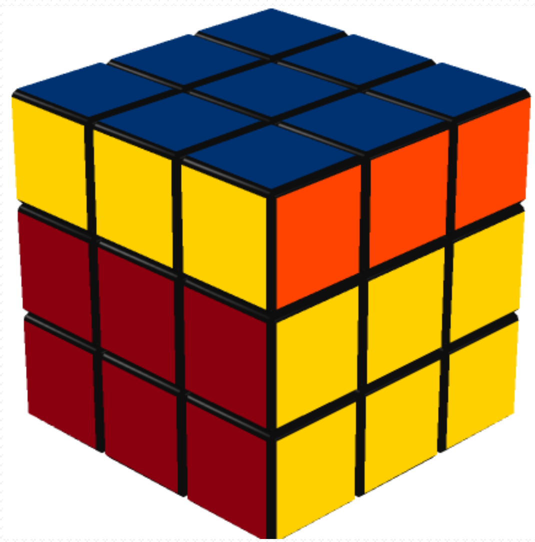
Cubies

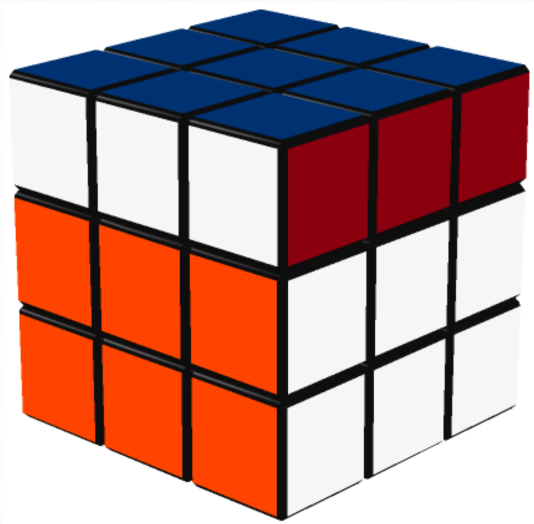
The images below correspond to an **upper or U twist** on the Rubik's cube
The U twist permutation has the form:

(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12)

↓

(4, 1, 2, 3, 5, 6, 7, 8, 9, 10 , 11, 12)





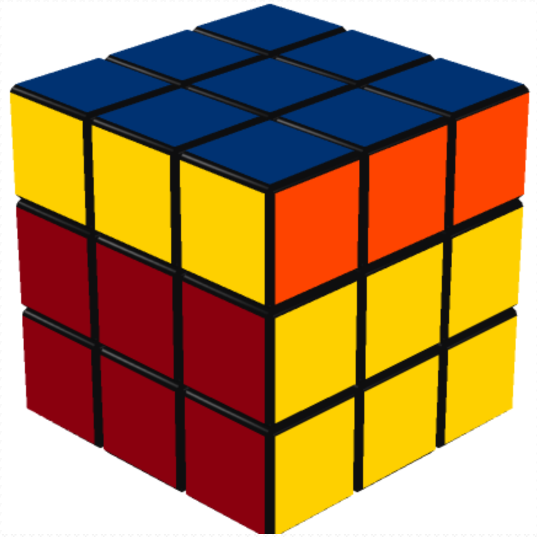
Cubies

We then take the permutation

$$(4, 1, 2, 3, 5, 6, 7, 8, 9, 10, 11, 12)$$

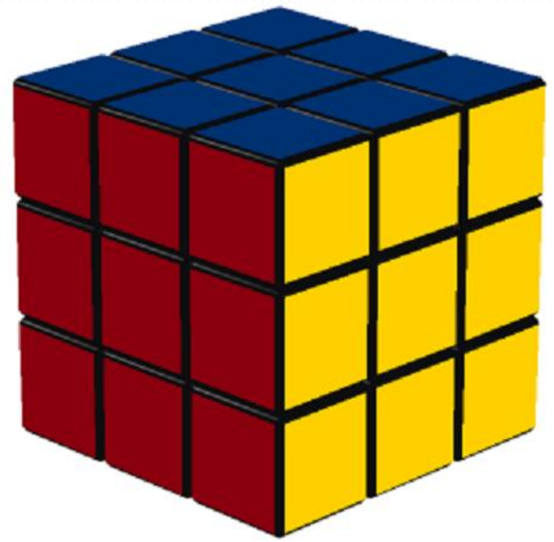
and express it in the **cyclic form**:

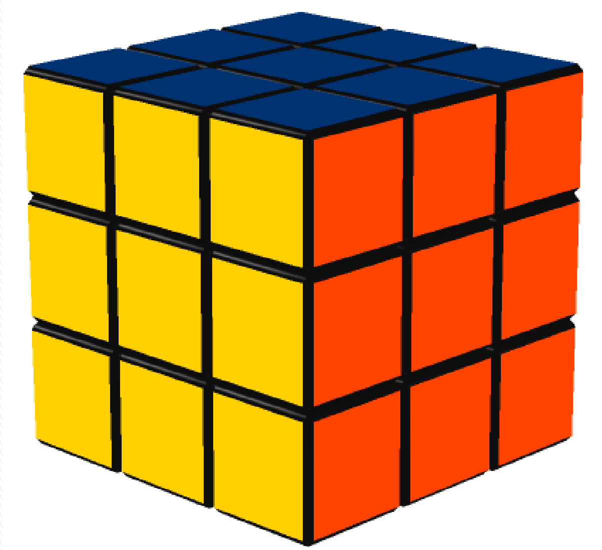
$$\tau = (1\ 2\ 3\ 4) = (12)(13)(14)$$



Orienting the Cubies

- We orient the faces of the corner cubies as such





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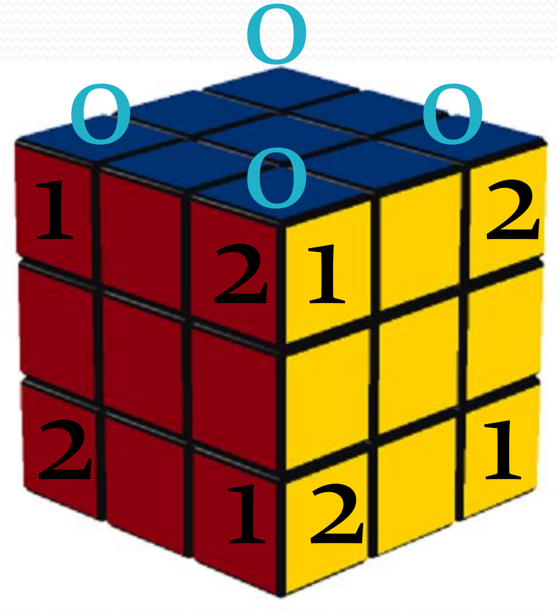
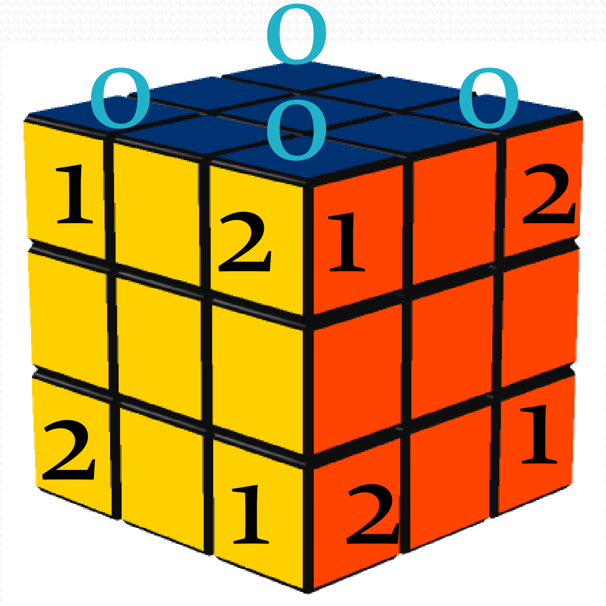
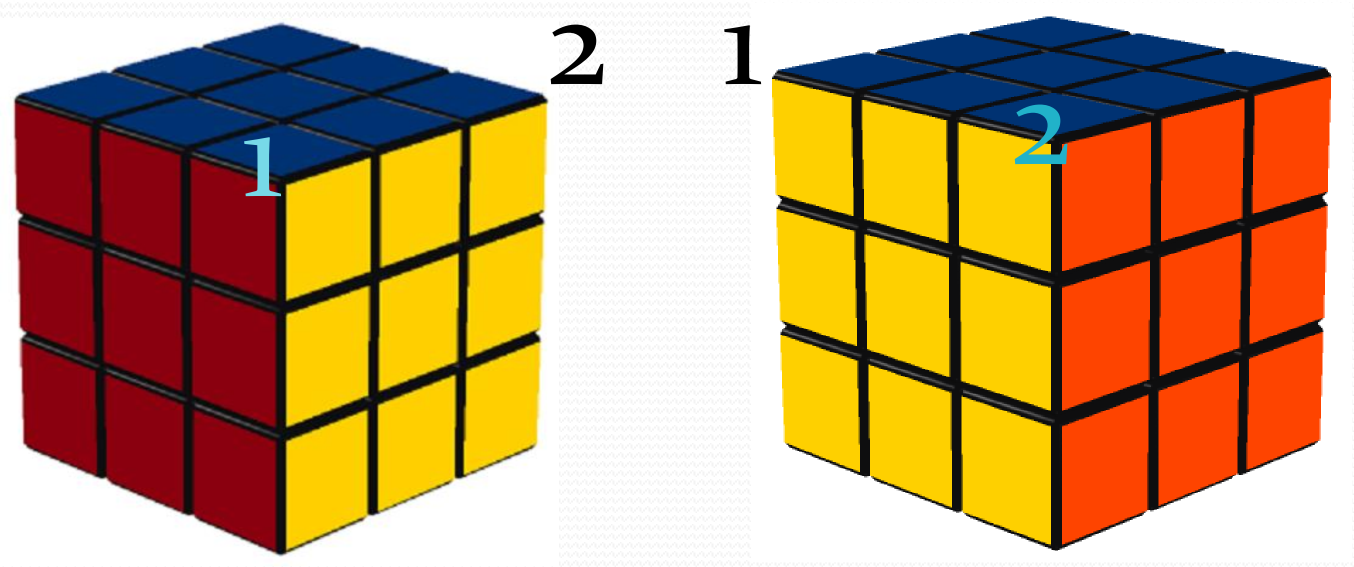




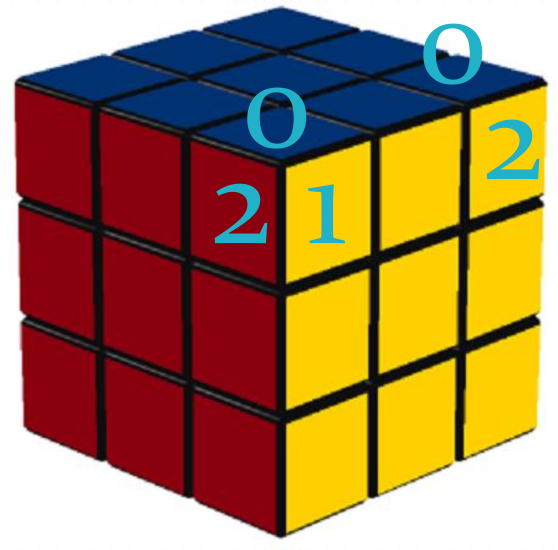
Orienting the Cubies

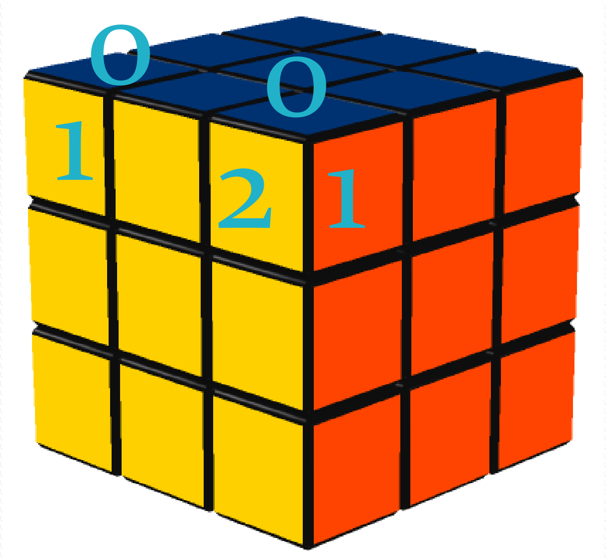
- Consider again the “1” and “2” corner cubies



Orienting the Cubies

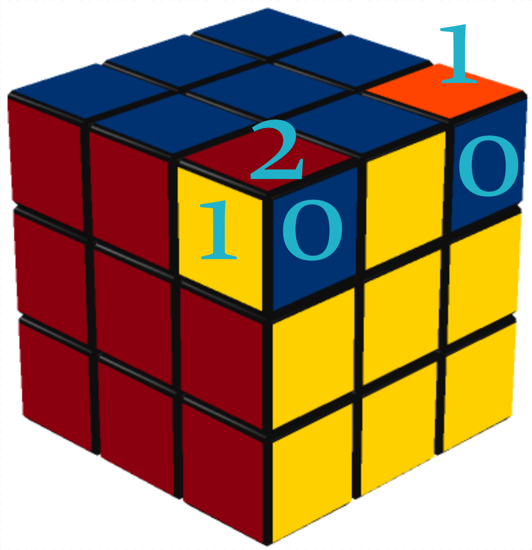
- Here is the orientation of 1 and 2

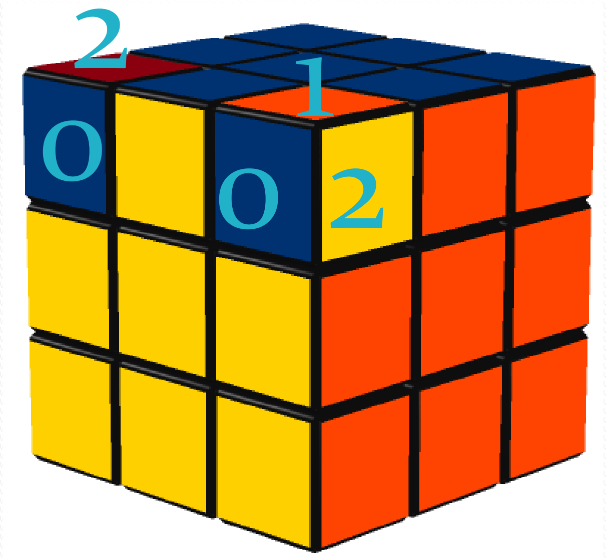




Orienting the Cubies

- If the cubies are oriented differently but not permutated, they will look like this:

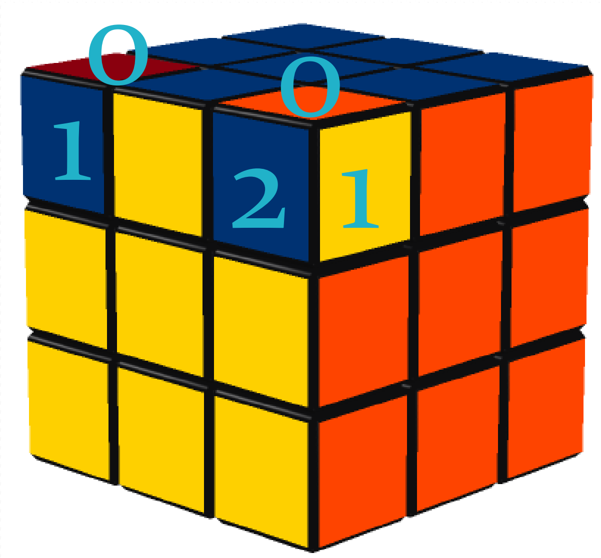




Orienting the Cubies

- BUT WAIT!

  The orientations of the cubicles are still the same



Orienting the Cubies

- When orienting the corner cubies we use the notation:

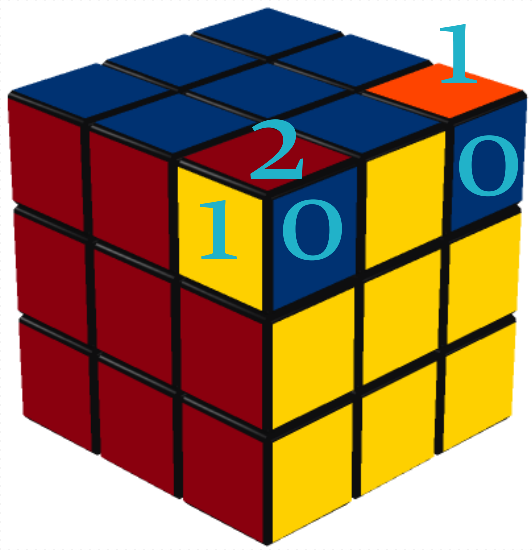
$$x = (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)$$

- The term, x_n represents the orientation number of the cubie in the **n** cubicle on the “o” cubicle face

Orienting the Cubies

- So for this example, our notation changes from this:

$$x = (0, 0, 0, 0, 0, 0, 0, 0)$$



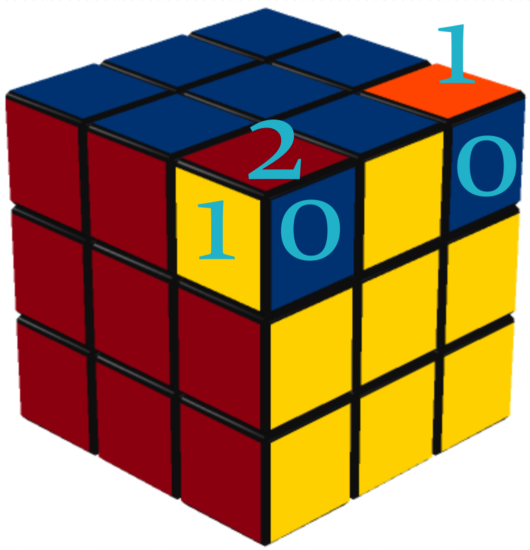
Cubie

Cubicle

Orienting the Cubies

- To this:

$$x = (2, 1, 0, 0, 0, 0, 0, 0)$$



Cubie

Cubicle

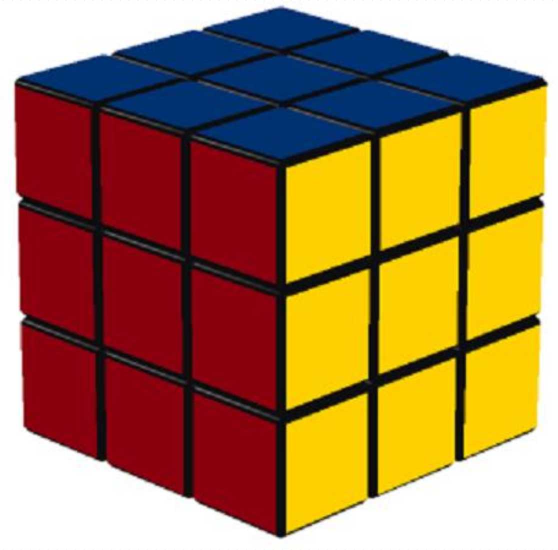
Orienting the Cubies

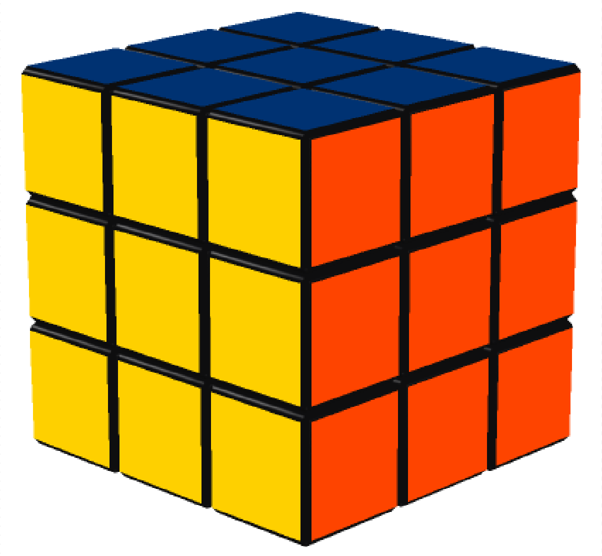
- When a U twist is applied, the edge orientations are unaffected

$$x = (0, 0, 0, 0, 0, 0, 0, 0)$$

Orienting the Cubies

- Similarly, we orient the edge cubies like so





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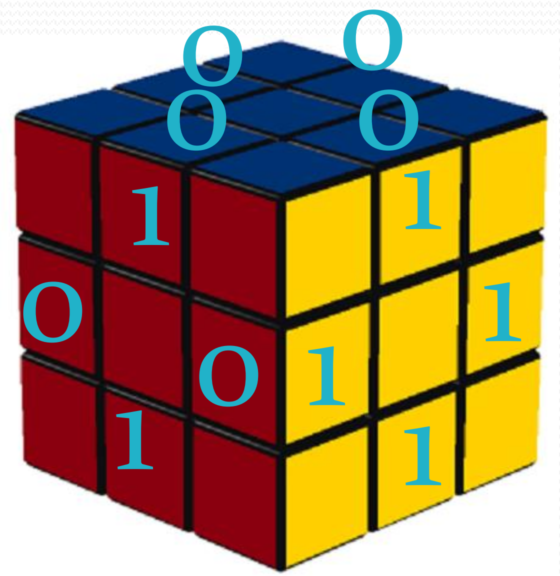
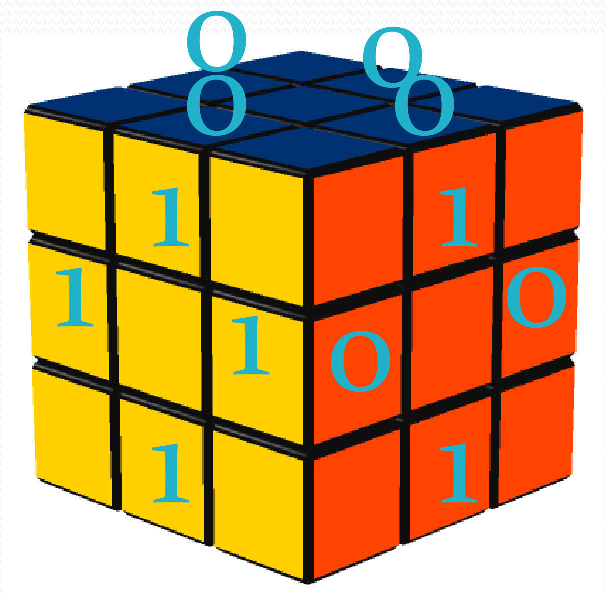
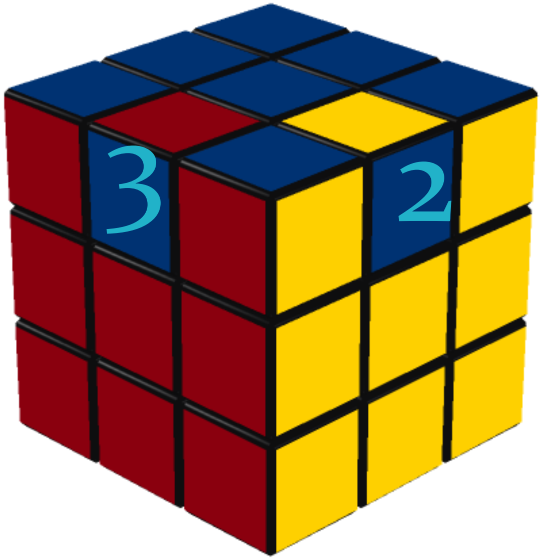


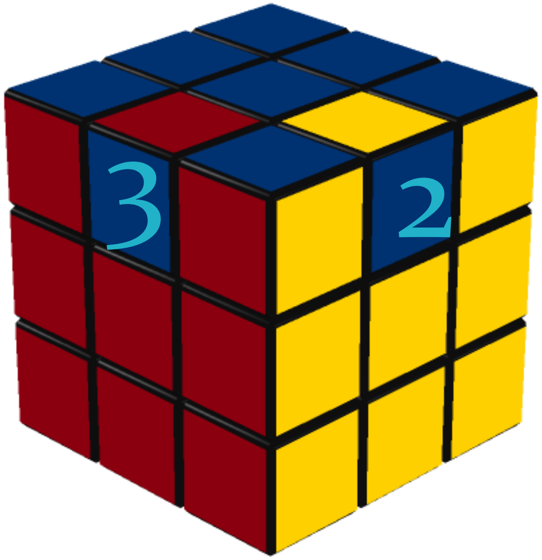


Orienting the Cubies

- Now consider this orientation of edge cubies “2” and “3”



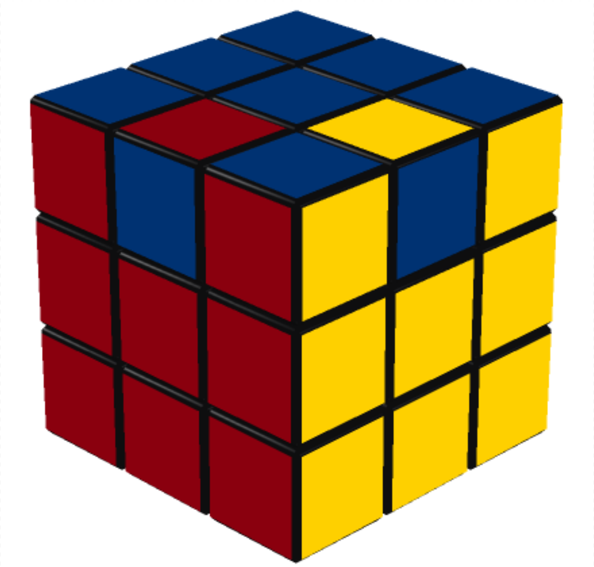
Cubie



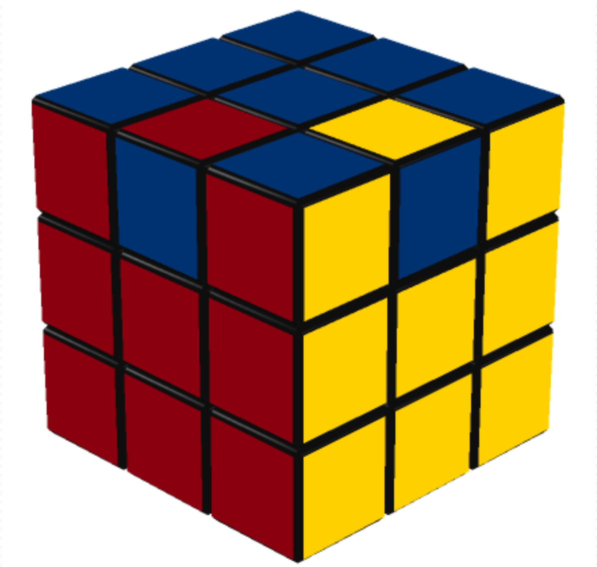
Cubicle

Orienting the Cubies

- Their number orientations are as follows:



Cubie



Cubicle

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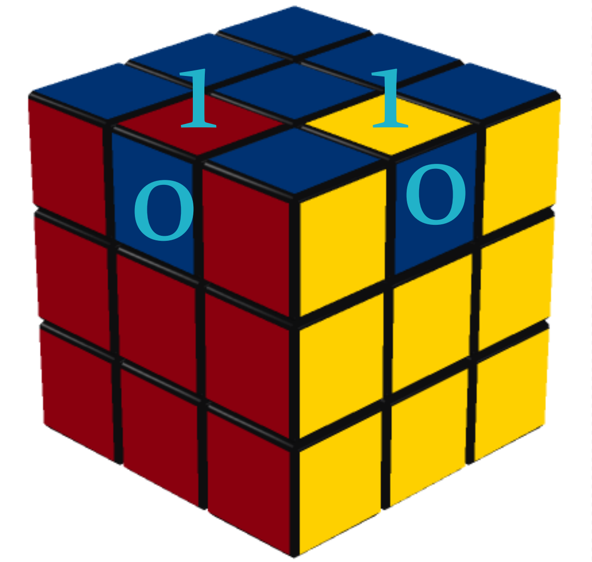
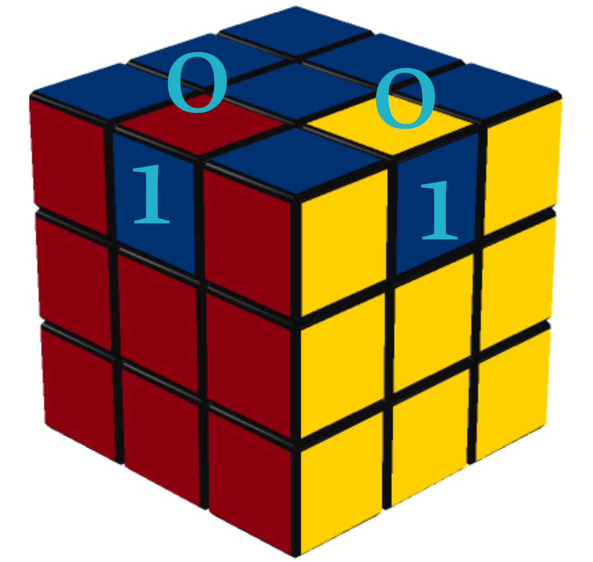


Cubie



Cubicle

Orienting the Cubies

- When orienting the edge cubies we use the notation:

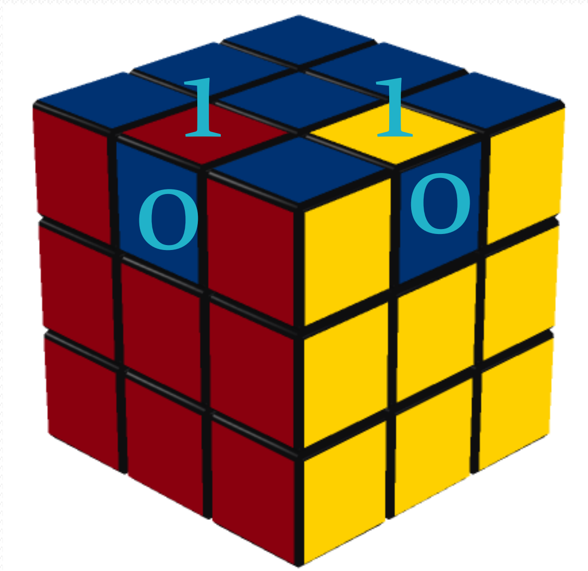
$$y = (y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9, y_{10}, y_{11}, y_{12})$$

- The term, y_n represents the orientation number of the cubie in the **n** cubicle on the “0” cubicle face

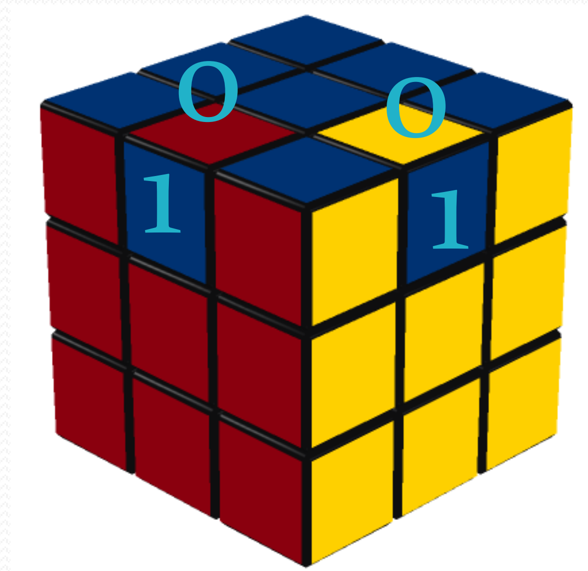
Orienting the Cubies

- So our notation changes from this:

$$y = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$$



Cubie

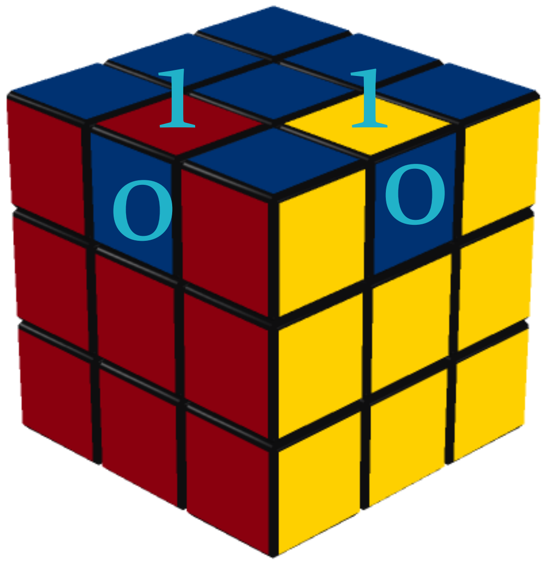


Cubicle

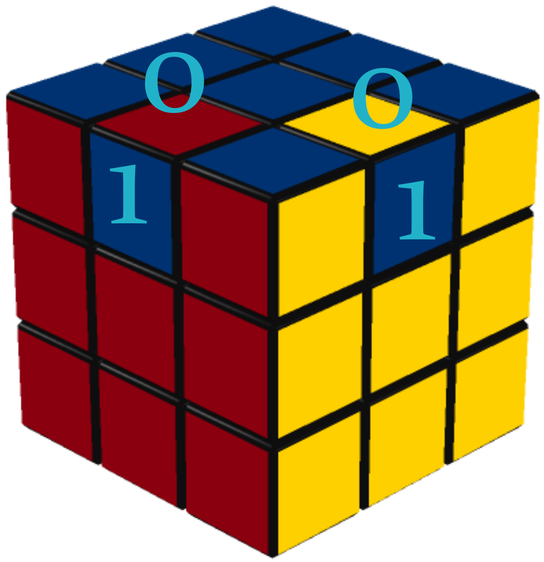
Orienting the Cubies

- To this:

$$y = (0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0)$$



Cubie

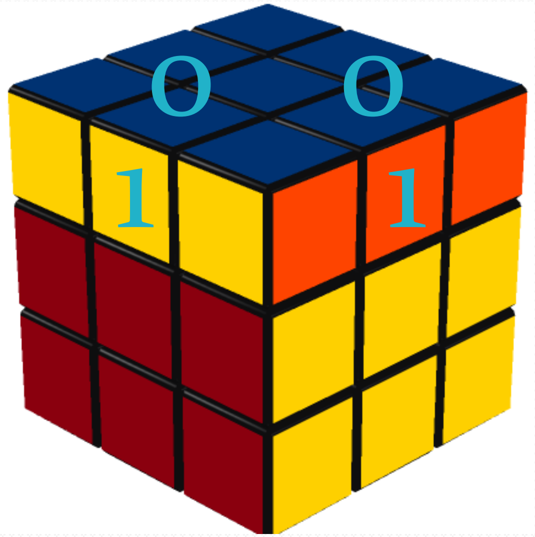


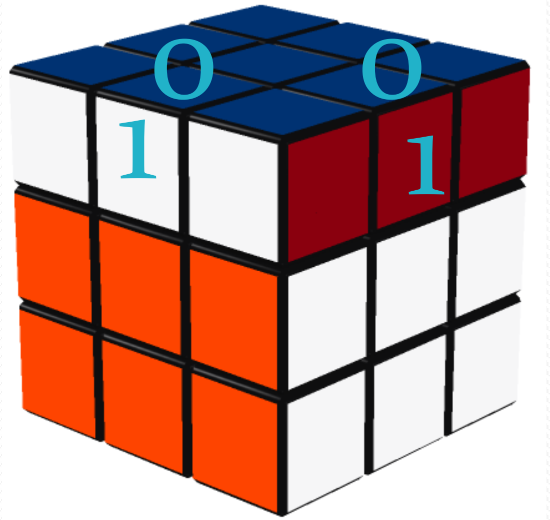
Cubicle

Orienting the Cubies

- Note that when a U twist is applied, the edge orientation is unaffected

$$y = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$$





Configurations

- We can express the U twist with the form (σ, τ, x, y)

$\sigma = (1\ 4\ 3\ 2)$

$\tau = (1\ 2\ 3\ 4)$

$x = (0, 0, 0, 0, 0, 0, 0, 0)$

$y = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$

OR

[(1 4 3 2), (1 2 3 4), (0, 0, 0, 0, 0, 0, 0, 0), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)]

Configurations

- All configurations of the cube can be expressed in this form.

- For example the move M = U L R U can be expressed as
- [(1 4 8 7)(2 6 5 3),(1 6 10 7 3 8 12 5)(2 4),
(1, 2, 1, 2, 2, 1, 2, 1,),(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,)]

Extracting the details

- Every twist on the Rubik's cube preserves the sum of both the edge cubies and the corner cubies.

$$\Sigma x_i = 0(\text{mod}3)$$

$$\Sigma y_i = 0(\text{mod}2)$$

- There is an agreement of either even or odd 2-cycles between σ and τ.
- This can be expressed through a sign homomorphism as

$$\text{sgn}(\sigma)=\text{sgn}(\tau),$$

where

sgn(even 2-cycles) = 1; sgn(odd 2-cycles) = -1

What this means

- First we define a valid configuration to be a (σ, τ, x, y) form where (σ, τ, x, y) can be attained by moves on the solved Rubik's cube.

- Thus we have that if a configuration is valid, then

$$\operatorname{sgn}\sigma = \operatorname{sgn}\tau$$

$$\Sigma x_i = 0 \pmod 3$$

$$\Sigma y_i = 0 \pmod 2$$

Determining Validity

- Is the move U R L U valid?

  [(1 4 8 7)(2 6 5 3),(1 6 10 7 3 8 12 5)(2 4),

  (1, 2, 1, 2, 2, 1, 2, 1,),(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,)],

Determining Validity

- Is the move U R L U valid?

  [(1 4 8 7)(2 6 5 3),(1 6 10 7 3 8 12 5)(2 4),

  (1, 2, 1, 2, 2, 1, 2, 1,),(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,)],

It is **valid** because

σ = (1 4 8 7)(2 6 5 3)=(1 4)(1 8)(1 7)(2 6)(2 5)(2 3)

τ = (1 6 10 7 3 8 12 5)(2 4)=(1 6)(1 10)(1 7)(1 3)(1 8)(1 12)(1 5)(2 4)

$\Sigma x_i = 1 + 2 + 1 + 2 + 2 + 1 + 2 + 1 = 12 = 0 \bmod 3$

$\Sigma y_i = 0 + \ldots + 0 = 0 \bmod 2$

$\mathrm{sgn}(\sigma) = 1 = \mathrm{sgn}(\tau)$

An Invalid Configuration

- Consider the form

[(1 4 8 7)(2 6 5 3),(1 6 10 7 3 8 12 5)(2 4),
(1, 2, 1, 2, 2, 1, 2, 1,),(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, **1**,)]

- This is similar to the move U L R U, except the y_{12} component has changed to 1
- This means $\Sigma y_i = 1 = 1 \bmod 2$
- Since $\Sigma y_i \neq 0 \bmod 2$ this is an **invalid** configuration

The Theorem

A configuration of the form(σ,τ,x,y) is valid *if and only if* $\text{sgn}\sigma = \text{sgn}\tau$, $\Sigma x_i = 0(\text{mod}3)$, and $\Sigma y_i = 0(\text{mod}2)$

Number of Configurations

- From this theorem we can uncover the total possible number of valid configurations
- The possibilities come from:
  - 8! From permuting the corner cubies
  - 12! From permuting the edge cubies
  - 3^8 From orienting the corner cubies
  - 2^{12} From orienting the edge cubies
- So that's

  $(8! \times 12! \times 3^8 \times 2^{12}) \approx$ 519 quintillion or 5.19×10^{20}

Number of Configurations

- But this number represents all of the valid and invalid configurations.
- To find the number of valid configurations:
  - Divide by 3 – only 1/3 of the possible corner orientations add to 0(mod3)
  - Divide by 2 – only ½ of the possible edge orientations add to 0(mod2)
  - Divide by 2 – only ½ of the possible corner and edge permutations agree on sign.

Number of Configurations

- So this number becomes

  $(8! \times 12! \times 2^{12} \times 3^{8}) / 12 = 43252003274489856000$

References

- Janet Chen. Group Theory and The Rubik's Cube. <http://www.math.harvard.edu/~jjchen/docs/Group%20Theory%20and%20the%20Rubik's%20Cube.pdf>.

- Images taken from the software "CubeTwister" created by Werner RandelShofer, <http://www.randelshofer.ch/cubetwister/>.

Questions