Rearrangement Groups of Fractals

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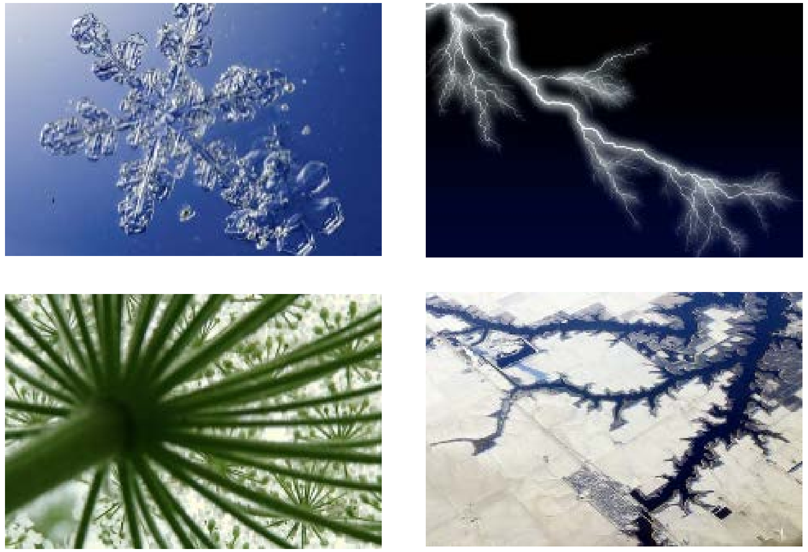
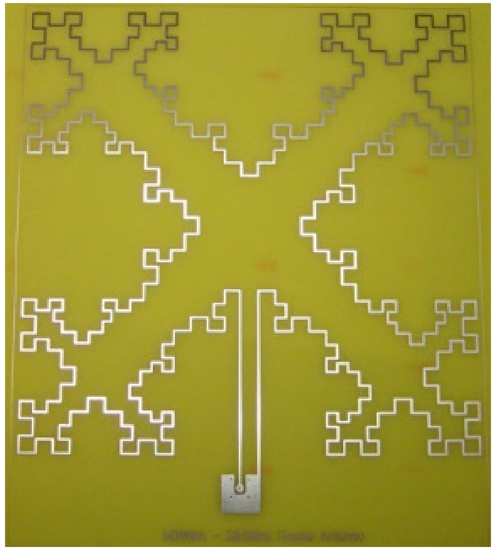
What is a Fractal?

- Geometric figure that is infinitely self-similar
- Found throughout nature
- Used in technology such as antennas to cover more area and respond to more frequencies than simple 1-dimensional lines

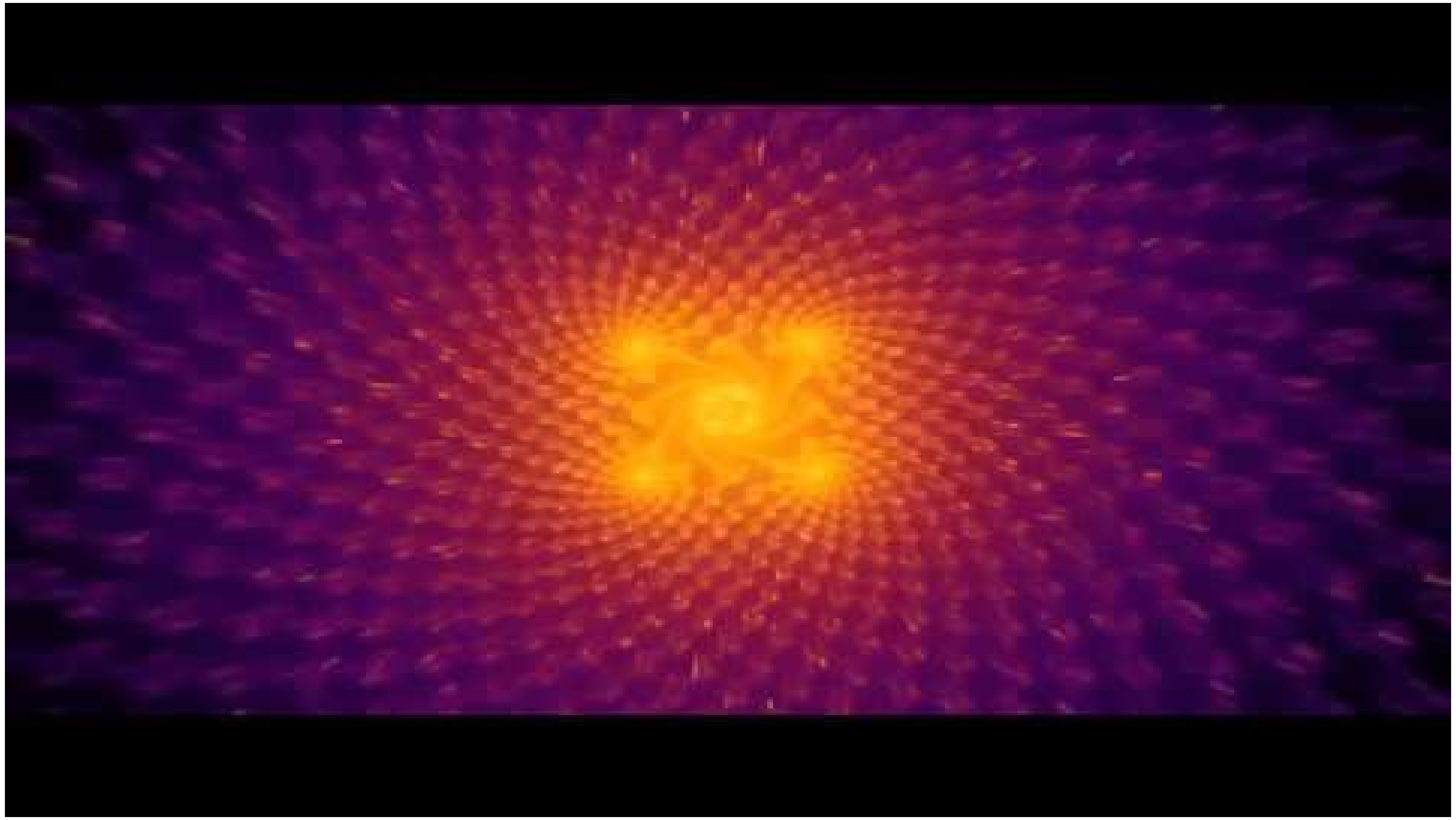


Fractals in Nature



Fractal Antenna

Self-Similarity



Groups

- Set G with operation * such that:
  - G is **Closed** → If $a,b \in G$, then $a*b \in G$
  - G is **Associative** → $a*(b*c)=(a*b)*c$
  - G has an **Identity** → There exists $e \in G$ such that $a*e=e*a=a$
  - All elements are **Invertible** → For all $a \in G$, there exists a^{-1} such that $a*a^{-1}=a^{-1}*a=e$

Symmetric Group

- S_4=permutations of {1,2,3,4}
- 6 basic transpositions to make S_4:

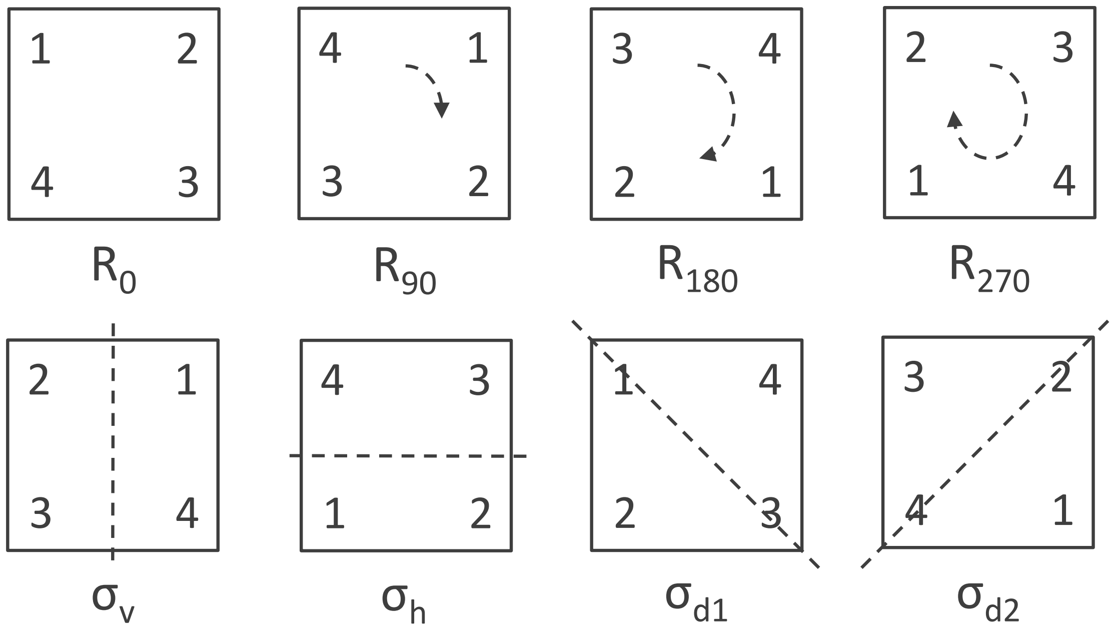
(12)=2134	(23)=1324
(13)=3214	(24)=1432
(14)=4231	(34)=1243

S_4 elements:

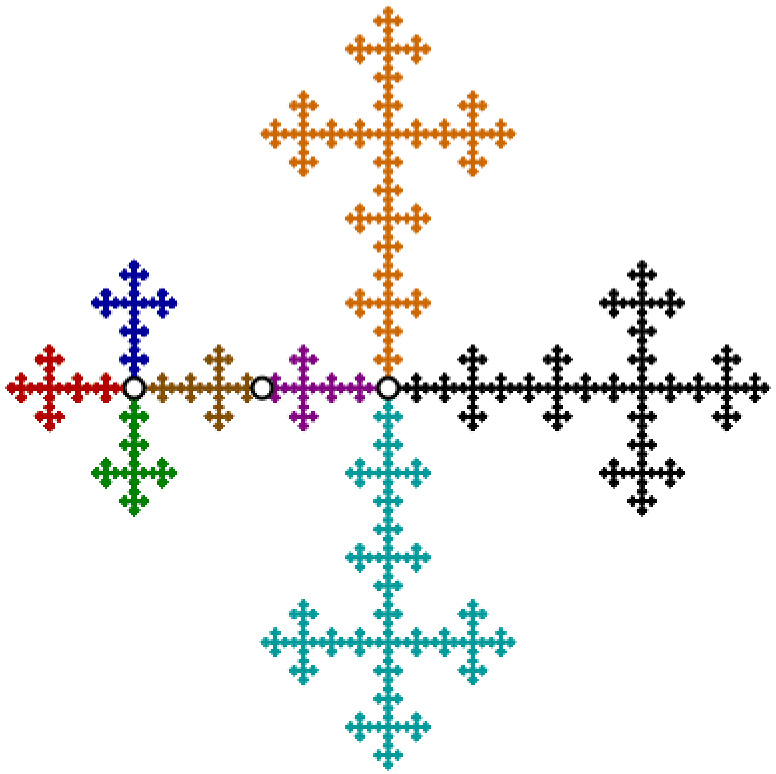
1234	2134	3124	4123
1243	2143	3142	4132
1324	2314	3214	4213
1342	2341	3241	4231
1423	2413	3412	4312
1432	2431	3421	4321

Dihedral Group

- D_4=symmetries of a square
- Subgroup of S_4



Fractal Symmetries



Generating Set

- Set of elements of *G* that can be multiplied together to produce all of *G*
- Generating set of S_4: (12), (1234)
  - 1→2, 2→1
  - 1→2, 2→3, 3→4, 4→1

Generating the Basic Transpositions

$(12)=(12)$

$(13)=(1234)^{-1}(12)(1234)^{2}(12)$

$(14)=(1234)^{-1}(12)(1234)$

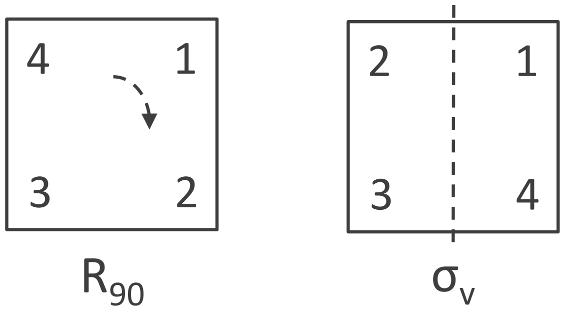
$(23)=(1234)^{-1}(12)(1234)$

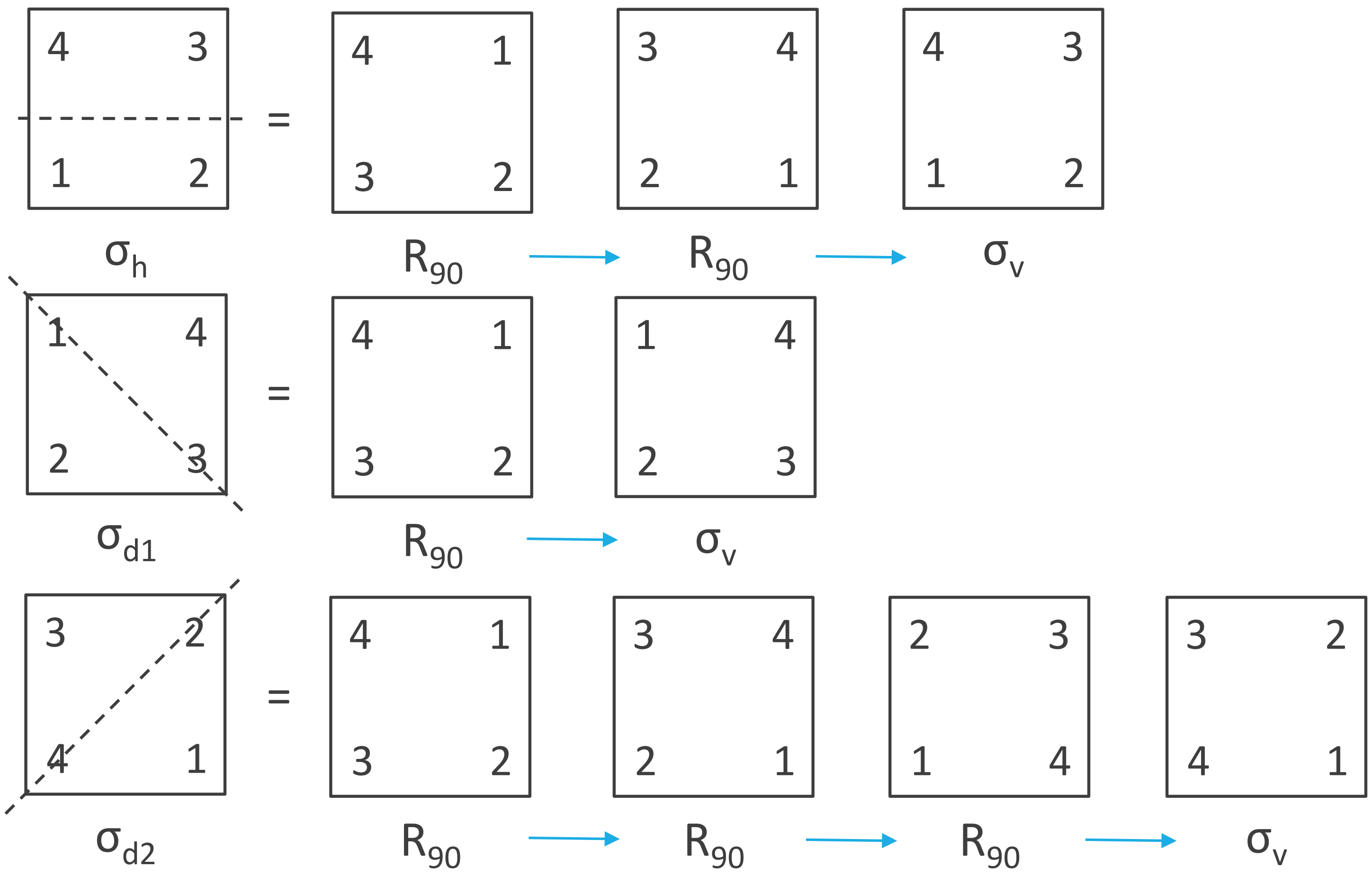
$(24)=(1234)^{2}(12)(1234)^{-2}(12)(1234)$

$(34)=(1234)(12)(1234)^{-1}(12)(1234)$

Generating Set

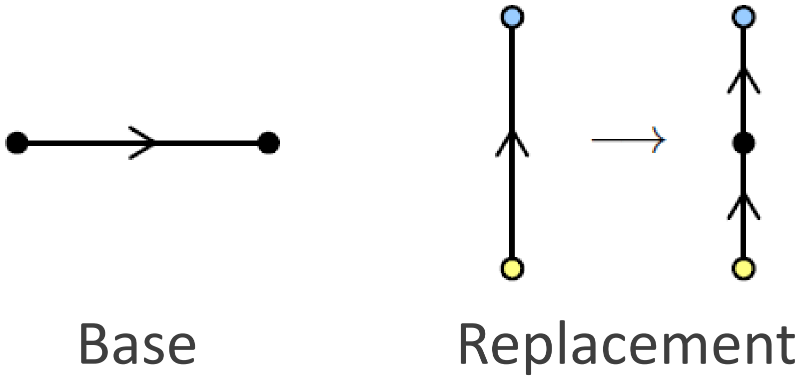
- Set of elements of *G* that can be multiplied together to produce all of *G*
- Generating set of D_4: R_{90}, σ_v





Replacement and Rearrangement Groups

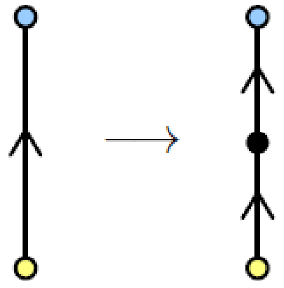
Replacement Rules



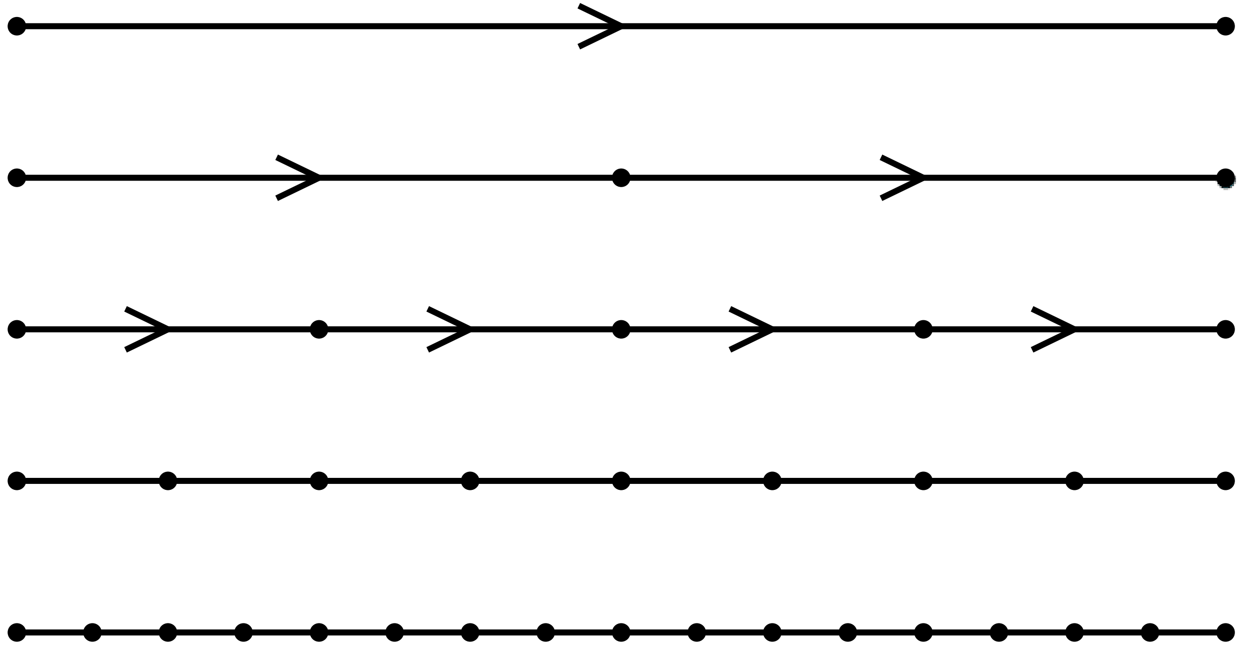
- Start with directed base graph G_0
- Expand by replacing each edge with given replacement rule
- Forms a group under composition
- Many but not all fractals can be generated this way

Example: Thompson’s Group F

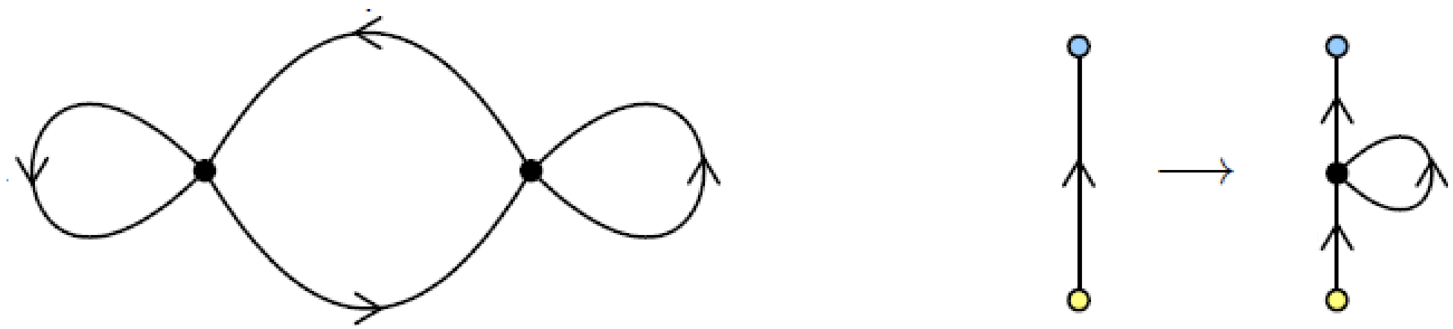
Base



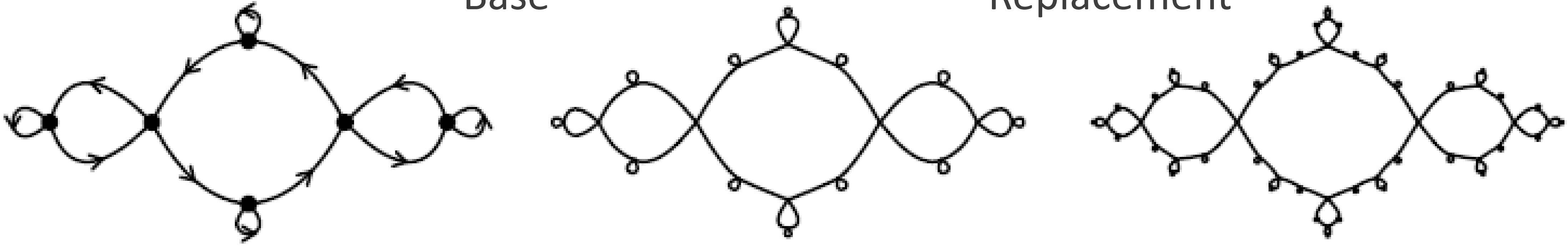
Replacement



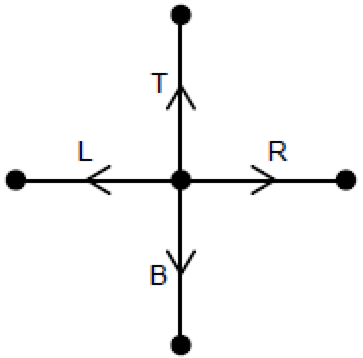
Example: Basilica



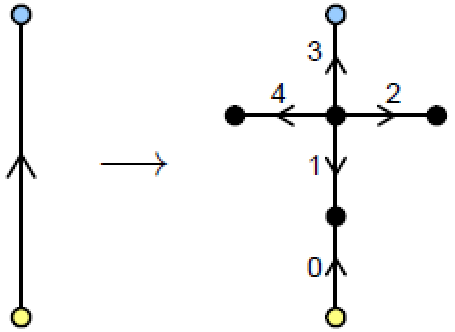
Base | Replacement



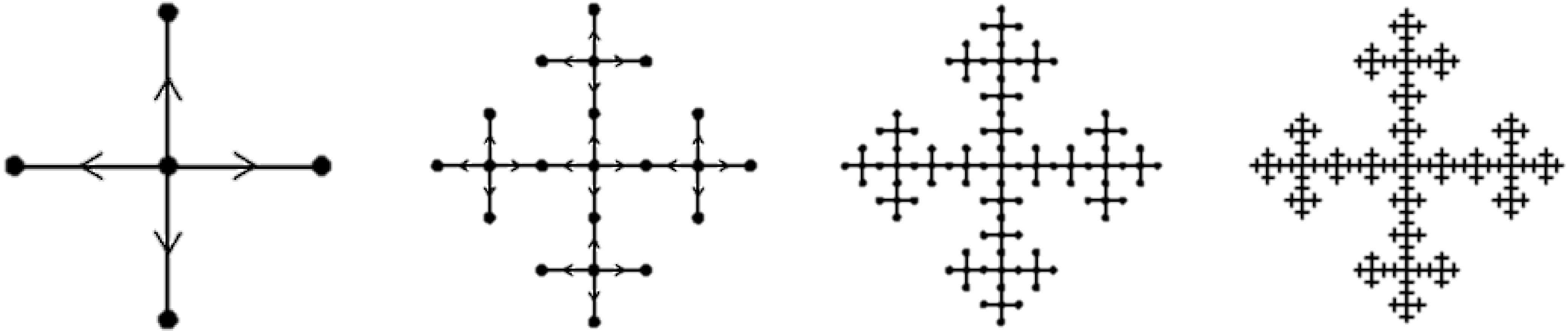
Example: Vicsek Fractal



Base



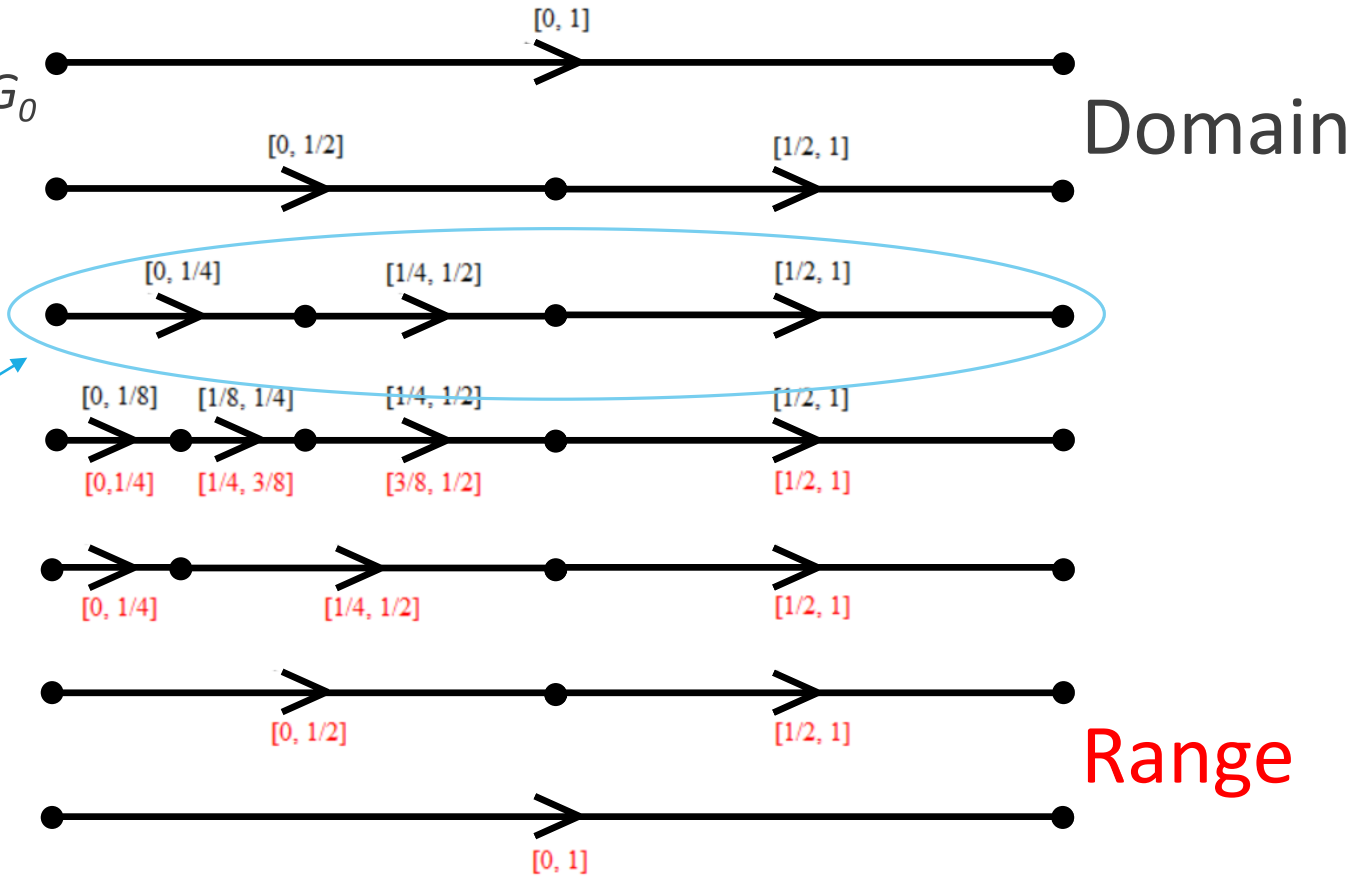
Replacement



Rearrangement Groups

- Definition:
  - An **expansion** is a graph obtained from G_0 by a finite number of applications of the replacement rule.

This is an expansion

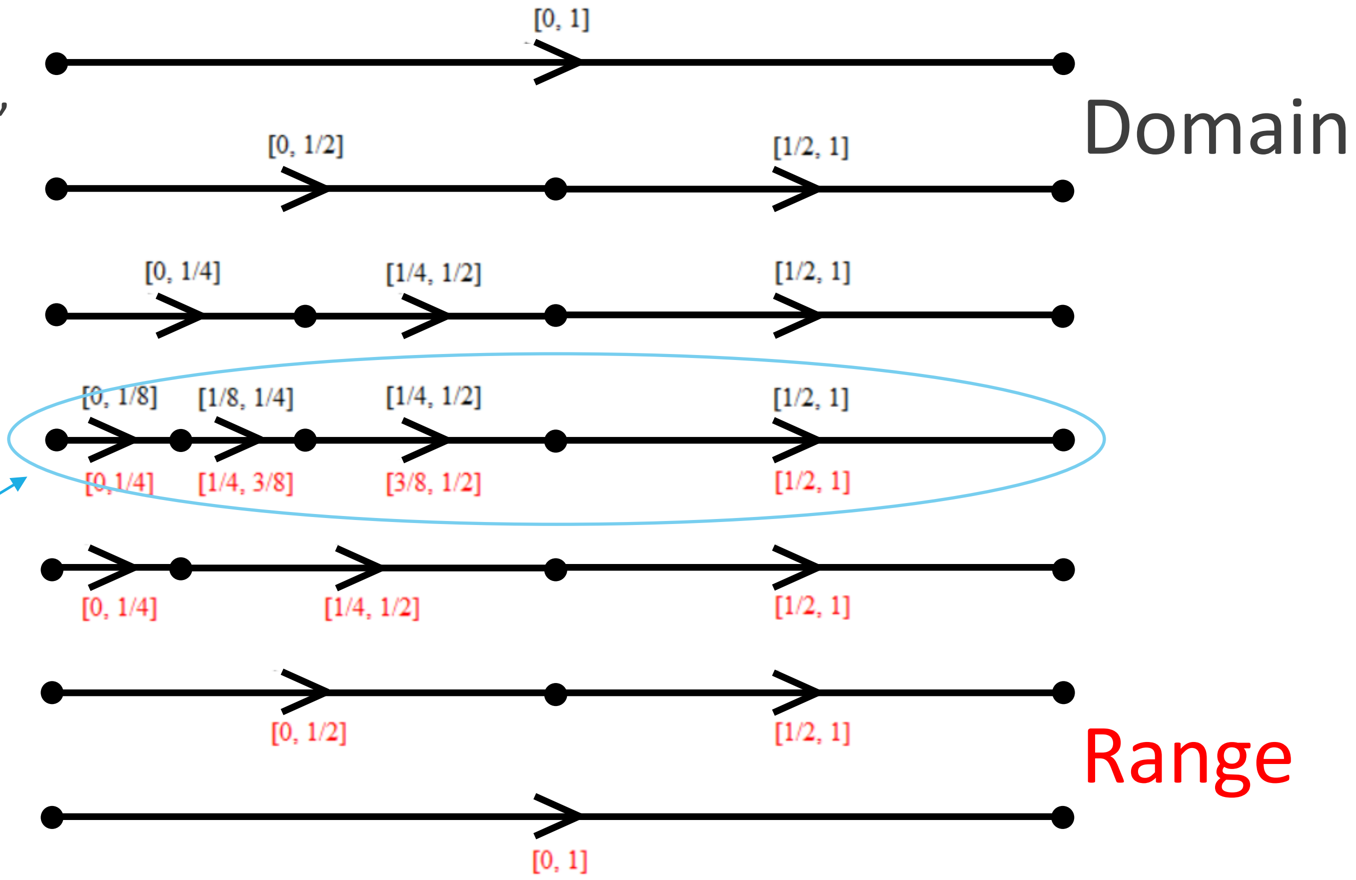


Rearrangement Groups

- Definition:
  - A **rearrangement** is a pair of expansions, *G* and *G'*, together with a graph isomorphism

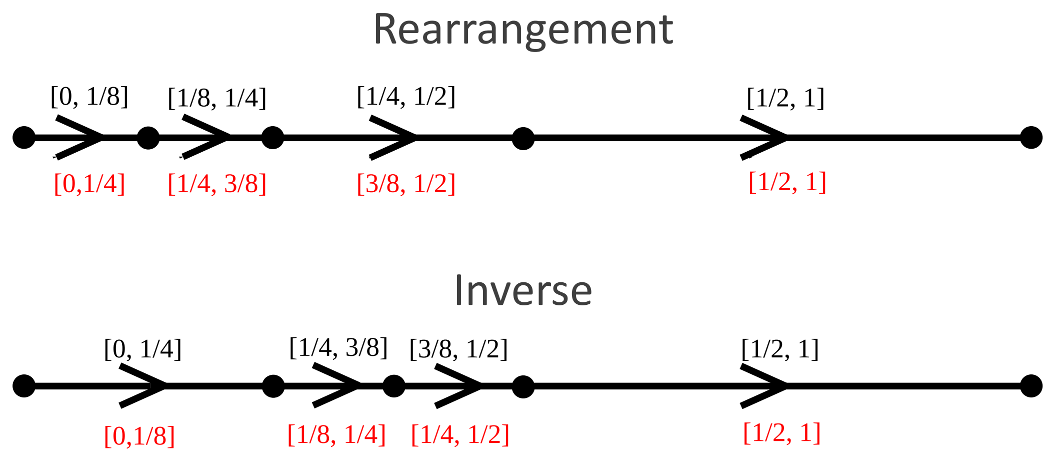
$$f: G \to G'$$

This is a rearrangement

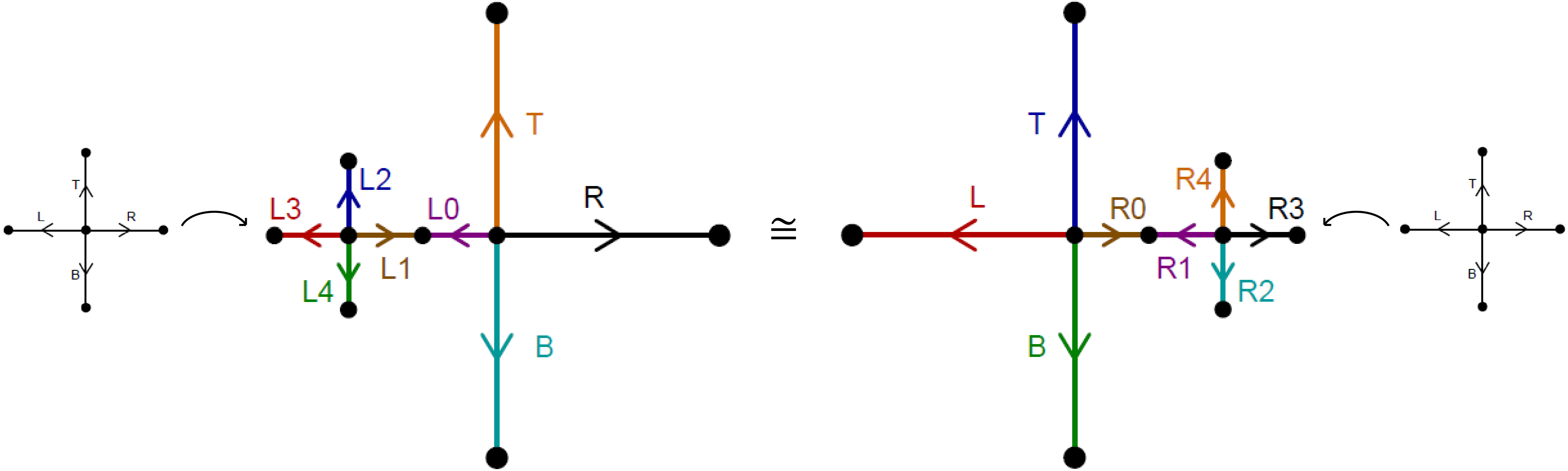


Rearrangement Groups

- Given a particular replacement system, the rearrangements form a group under composition
  - Two rearrangements make another rearrangement
  - Rearranging is associative
  - Identity is the base step
  - Inverse is the rearrangement in reverse



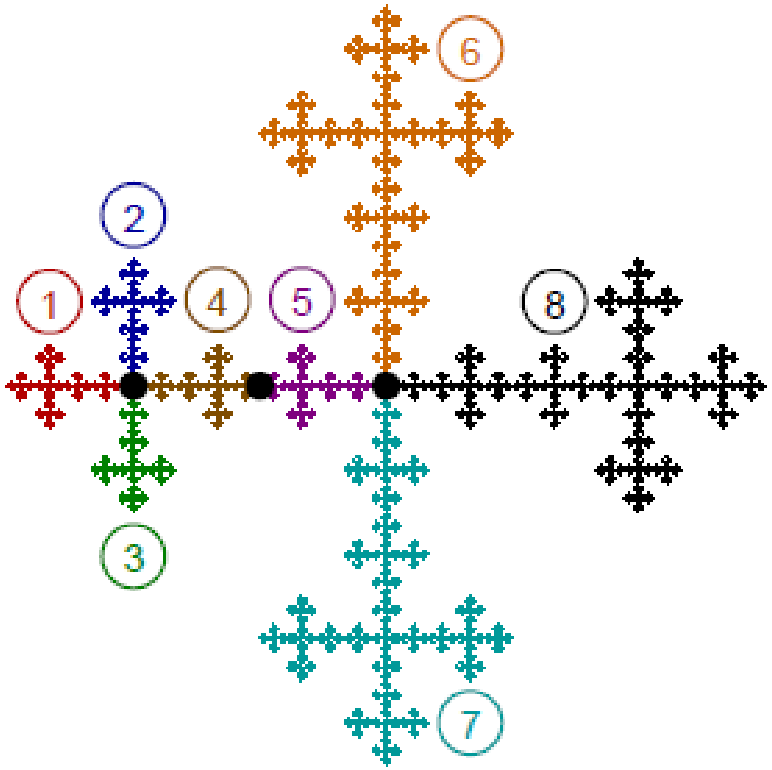
Vicsek Rearrangement



Domain

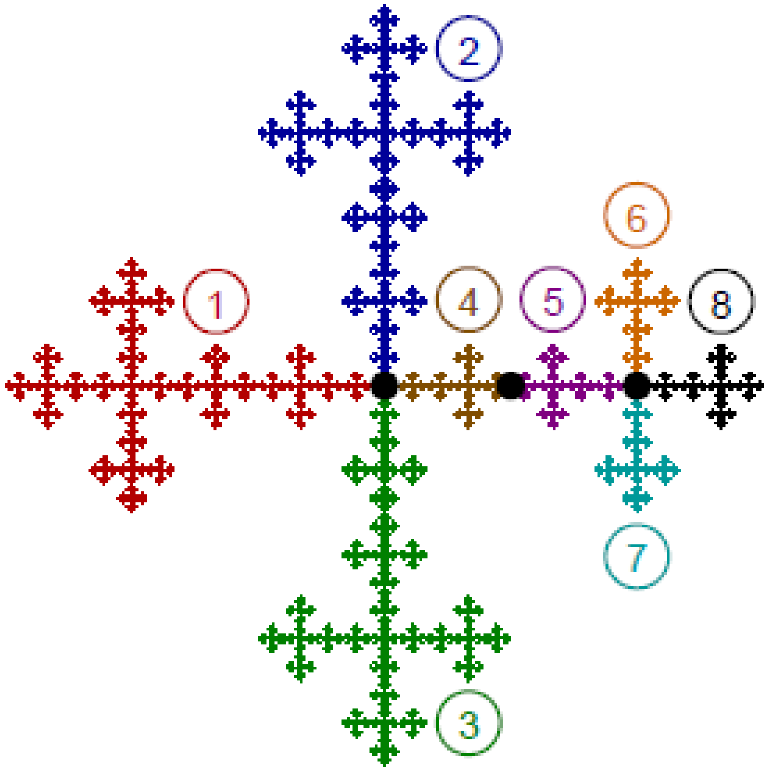
Range

Vicsek Rearrangement



Domain

→

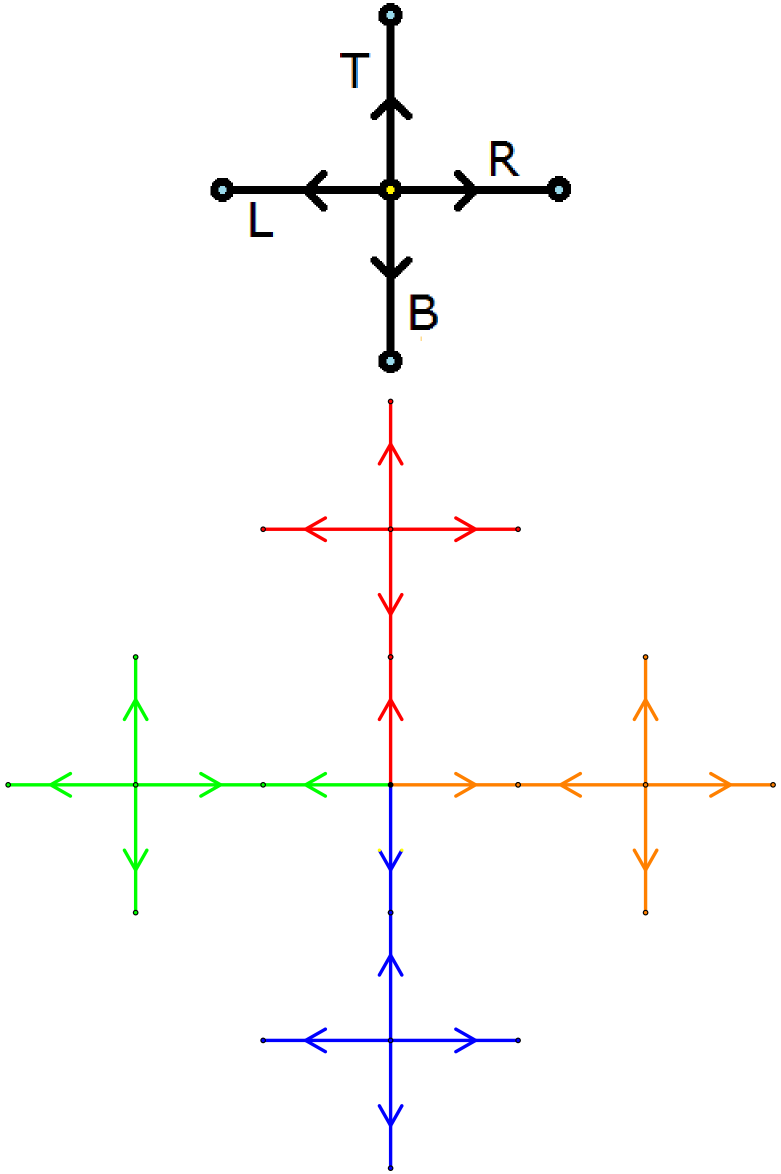


Range

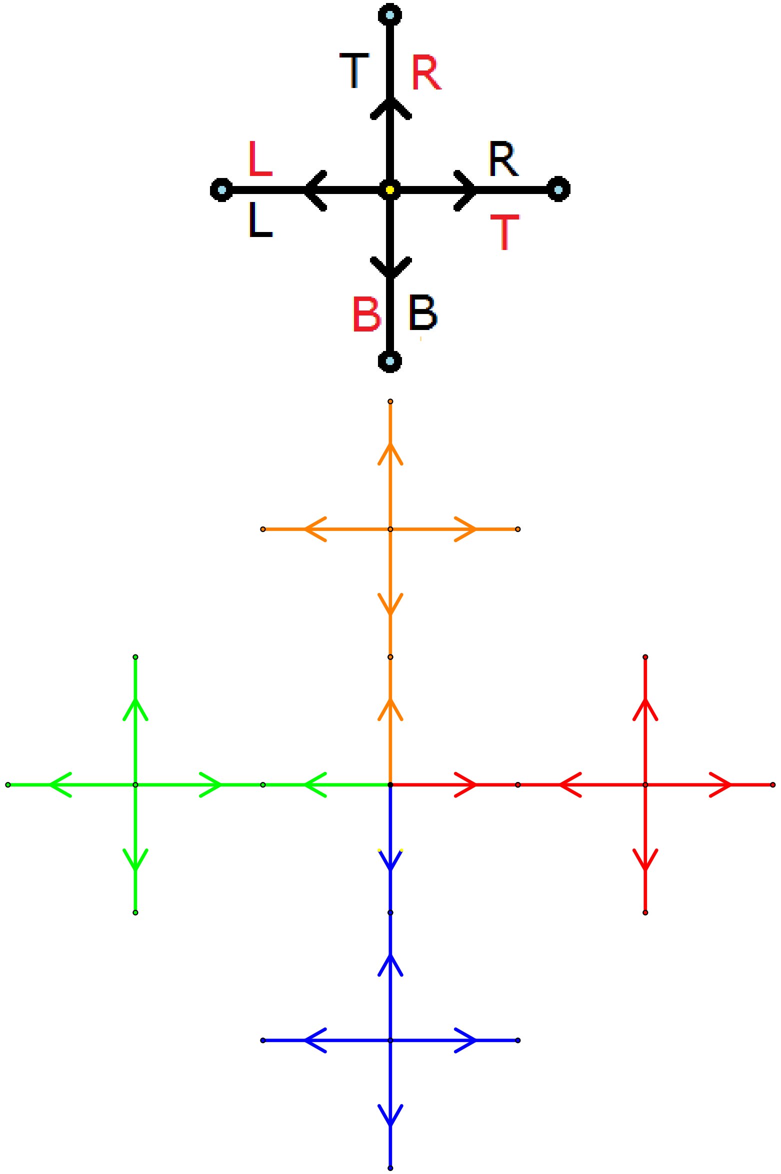
Goal

- Find a generating set for all possible rearrangements
- Hypothesis:
  - Set will correspond to S_4 and $F_{3,2}$

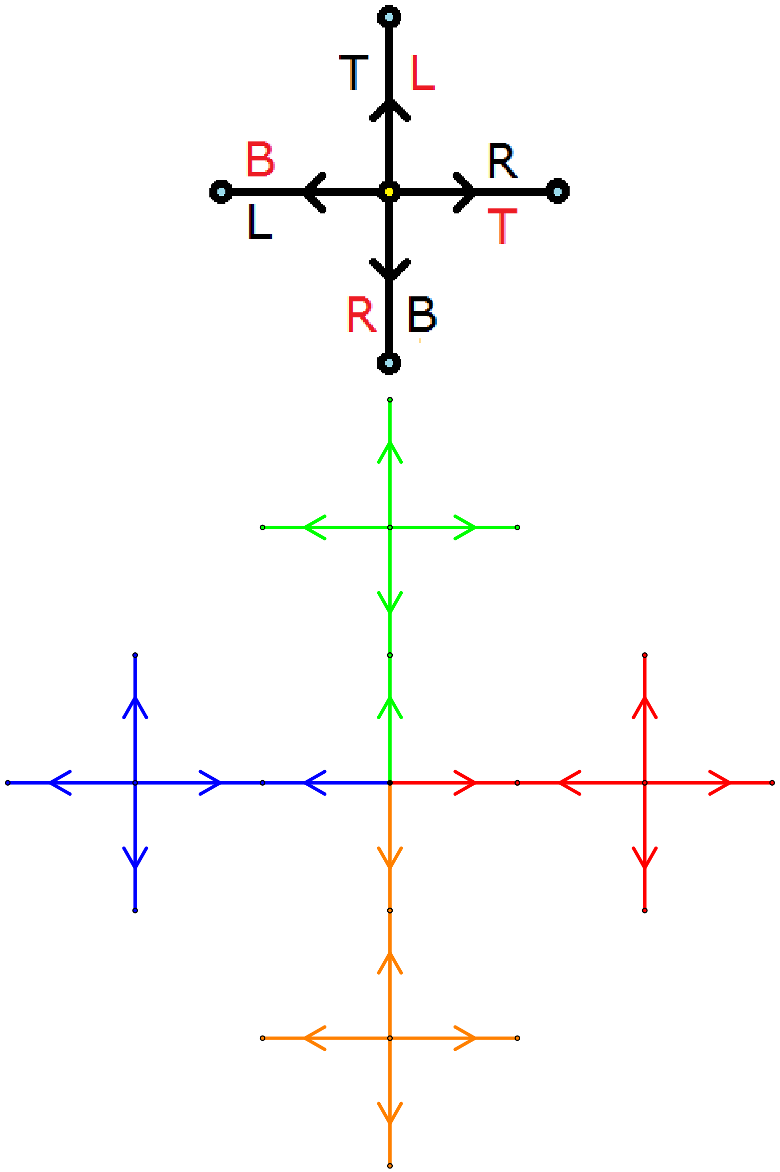
Base Step

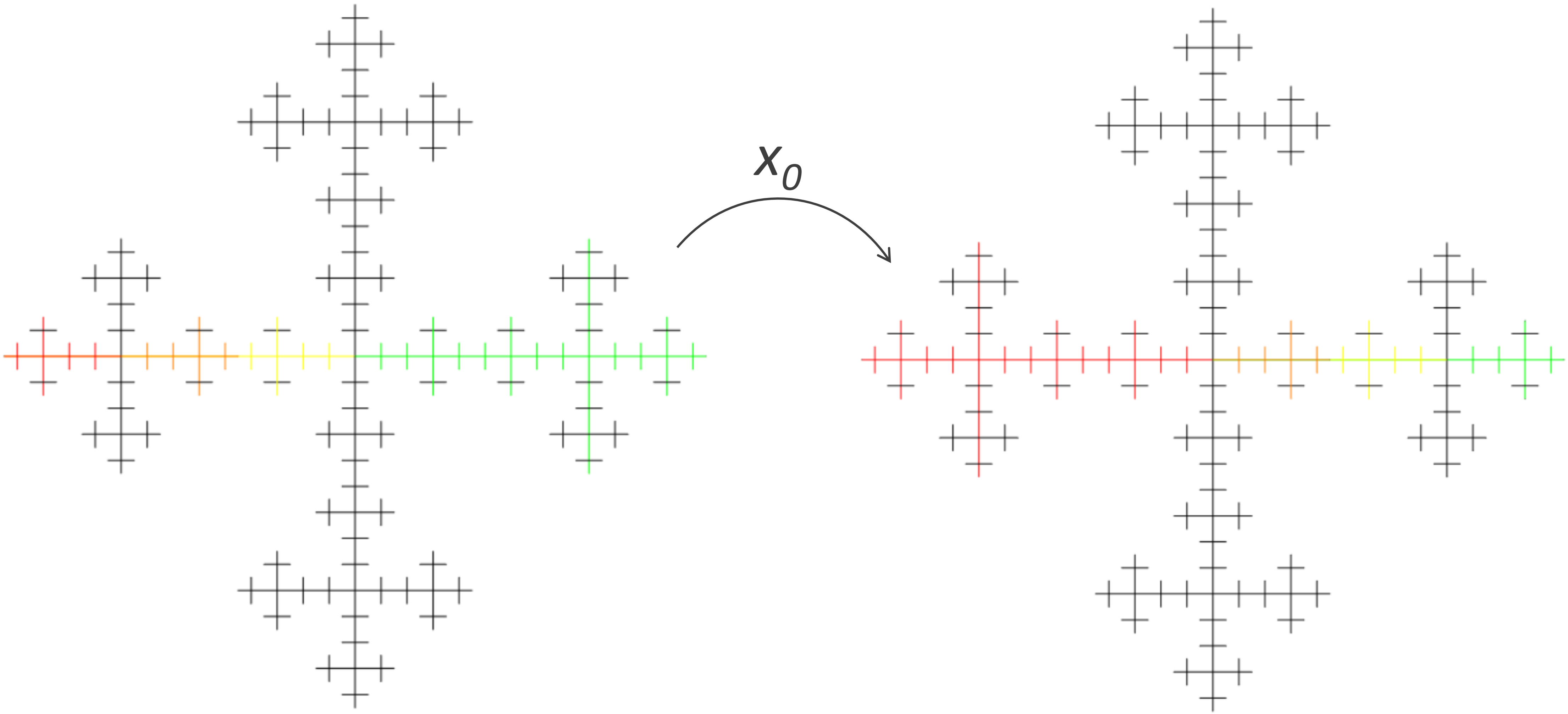


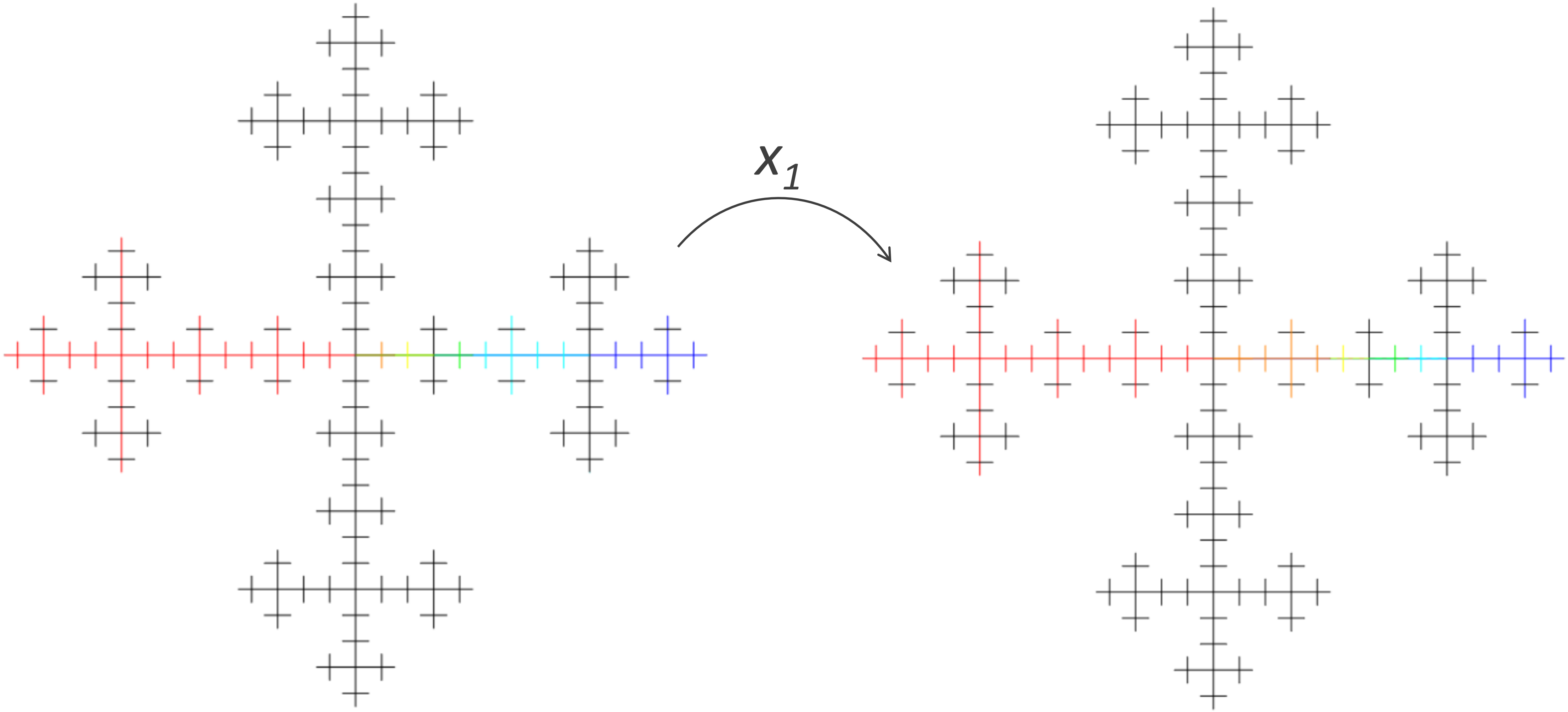
(12)

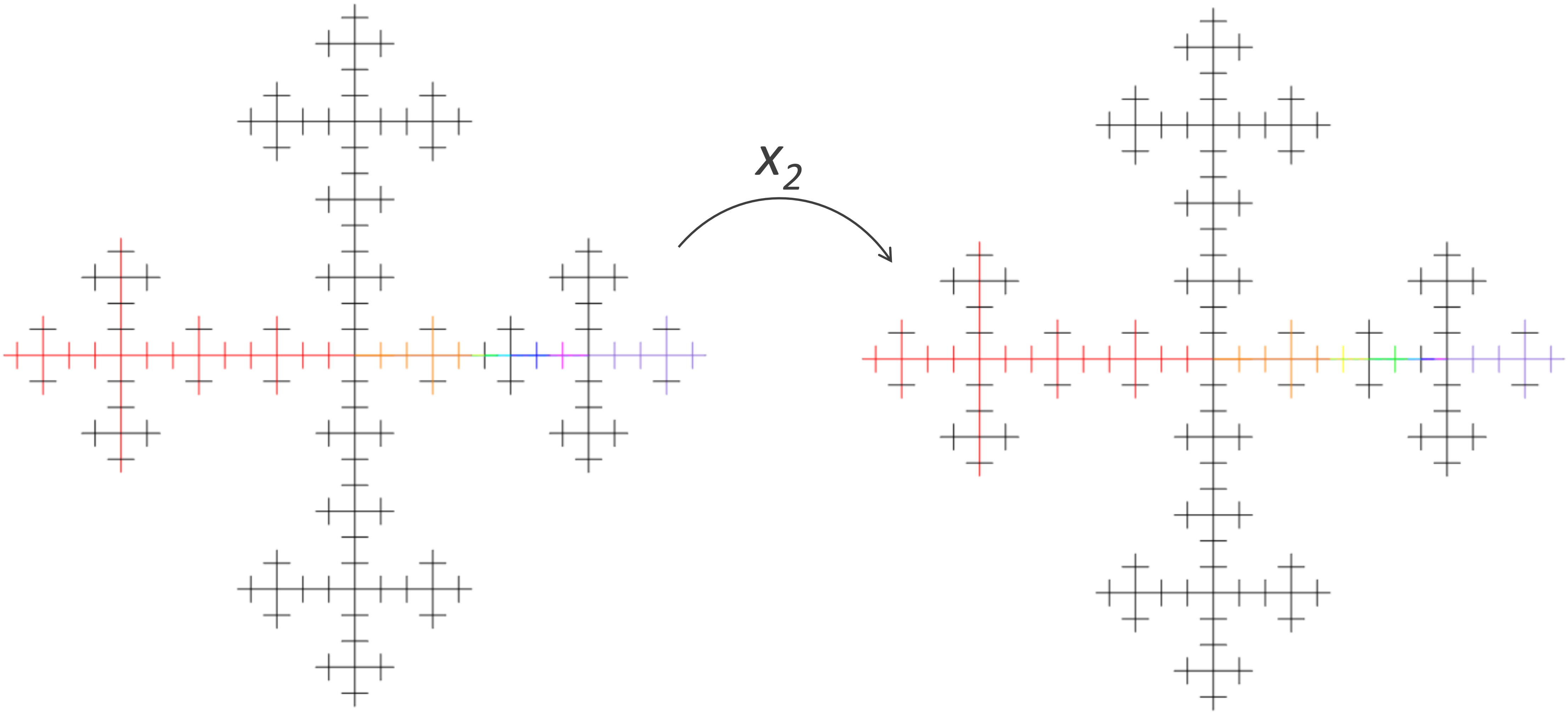


(1234)




x0


x_1


X_2